Diagrammatic Young projection operators for U (n)

Henriette Elvang, Predrag Cvitanović, and Anthony D. Kennedy

Citation: Journal of Mathematical Physics **46**, 043501 (2005); doi: 10.1063/1.1832753
View online: http://dx.doi.org/10.1063/1.1832753
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/46/4?ver=pdfcov
Published by the AIP Publishing

JOURNAL OF MATHEMATICAL PHYSICS **46**, 043501 (2005)

Diagrammatic Young projection operators for U(n)

Henriette Elvang
Department of Physics, UCSB, Santa Barbara, California 93106

Predrag Cvitanović
Center for Nonlinear Science, Georgia Institute of Technology, Atlanta, Georgia 30332-0430

Anthony D. Kennedy
School of Physics, JCMB, King's Buildings, University of Edinburgh, Edinburgh EH9 3JZ, Scotland

(Received 23 April 2004; accepted 12 October 2004; published online 14 March 2005)

We utilize a diagrammatic notation for invariant tensors to construct the Young projection operators for the irreducible representations of the unitary group U(n), prove their uniqueness, idempotency, and orthogonality, and rederive the formula for their dimensions. We show that all U(n) invariant scalars ($3n-j$ coefficients) can be constructed and evaluated diagrammatically from these U(n) Young projection operators. We prove that the values of all U(n) $3n-j$ coefficients are proportional to the dimension of the maximal representation in the coefficient, with the proportionality factor fully determined by its S_k symmetric group value. We also derive a family of new sum rules for the $3-j$ and $6-j$ coefficients, and discuss relations that follow from the negative dimensionality theorem. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1832753]

I. INTRODUCTION

Symmetries are beautiful, and theoretical physics is replete with them, but there comes a time when a calculation must be done. Innumerable calculations in high-energy physics, nuclear physics, atomic physics, and quantum chemistry require construction of irreducible many-particle states (irreps), decomposition of Kronecker products of such states into irreps, and evaluations of group theoretical weights (Wigner $3n-j$ symbols, reduced matrix elements, quantum field theory "vacuum bubbles"). At such times effective calculational methods gain in appreciation.

In his 1841 fundamental paper[1] on the determinants today known as "Jacobians," Jacobi initiated the theory of irreps of the symmetric group S_k. Schur used the S_k irreps to develop the representation theory of GL($n;\mathbb{C}$) in his 1901 dissertation,[2] and already by 1903 the Young tableaux[3,4] came into use as a powerful tool for reduction of both S_k and GL($n;\mathbb{C}$) representations. In quantum theory the group of choice[5] is the unitary group U(n), rather than the general linear group GL($n;\mathbb{C}$). Today this theory forms the core of the representation theory of both discrete and continuous groups, described in many excellent textbooks.[6–17]

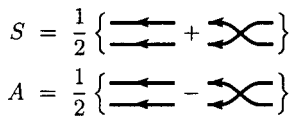
Here we transcribe the theory of the Young projection operators into a form particularly well suited to particle physics calculations, and show that the diagrammatic methods of Ref. 18 can be profitably employed in explicit construction of U(n) multiparticle states, and evaluation of the associated $3n-j$ coefficients.

In diagrammatic notation tensor objects are manipulated without any explicit indices. Diagrammatic evaluation rules are intuitive and relations between tensors can often be grasped visually. Take as an example the reduction of a two-index tensor T_{ij} into symmetric and antisymmetric parts, $T=(S+A)T$, where

$$ST_{ij} = \tfrac{1}{2}(\mathbb{I} + (12))T_{ij}$$

$$AT_{ij} = \tfrac{1}{2}(\mathbb{I} - (12))T_{ij},$$

and $\mathbb{I}$ and (12) denote the identity and the index transposition. Diagrammatically, the two projection operators are drawn as



(1)

It is clear at a glance that S symmetrizes and A antisymmetrizes the two tensor indices. Here we shall construct such projection operator for tensors of any rank.

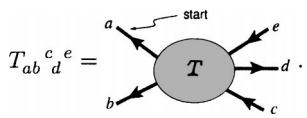
Penrose's papers are the first (known to the authors) to cast the Young projection operators into a diagrammatic form. Here we use Penrose diagrammatic notation for symmetrization operators,[19] Levi–Civita tensors[20] and "strand networks."[21] For several specific, few-particle examples, diagrammatic Young projection operators were constructed by Canning,[22] Mandula,[23] and Stedman.[24] A diagrammatic construction of the U(n) Young projection operators for *any* Young tableau was outlined in Ref. 25, without proofs. Here we present the method in detail, as well as the proof that the Young projection operators so constructed are unique.[26] The other new results are a proof that every U(n) $3n-j$ coefficient is proportional to the dimension of the largest irrep within the $3n-j$ diagram, and several sum rules for U(n) $3-j$ and $6-j$ coefficients.

The paper is organized as follows. The diagrammatic notation for tensors is reviewed in Sec. II and the Young tableaux in Sec. III. This material is standard and the reader is referred to any of the above cited monographs for further details. In Sec. IV we construct diagrammatic Young projection operators for U(n), and give formulas for the normalizations and the dimensions of U(n) irreps. In Sec. V we recast the Clebsch–Gordan recoupling relations into a diagrammatic form, and show that—somewhat surprisingly—the values of all U(n) $3n-j$ coefficients follow from the representation theory for the symmetric group S_k alone. The $3n-j$ coefficients for U(n) are constructed from the Young projection operators and evaluated by diagrammatic methods in Sec. V B. We derive a family of new sum rules for U(n) $3n-j$ coefficients in Sec. V C. In Sec. VI we briefly discuss the case of SU(n) and mixed multiparticle antiparticle states. In Sec. VII we state and prove the negative dimensionality theorem for U(n). Not only does this proof provide an example of the power of diagrammatic methods, but the theorem also simplifies certain group theoretic calculations. We summarize our results in Sec. VIII.

The key, but lengthy original result presented in this paper, the proof of the uniqueness, completeness, and orthogonality of the Young projection operators[26] is relegated to the appendix.

II. DIAGRAMMATIC NOTATION

In the diagrammatic notation[18] an invariant tensor is drawn as a "blob" with a leg representing each index. An arrow indicates whether it is an upper or lower index; lower index arrows always point away from the blob whereas upper index arrows point into the blob. The index legs are ordered in the counterclockwise direction around the blob, and if the indices are not cyclic there must be an indication of where to start, for example,



An internal line in a diagram implies a sum over the corresponding index: matrix multiplication is drawn as

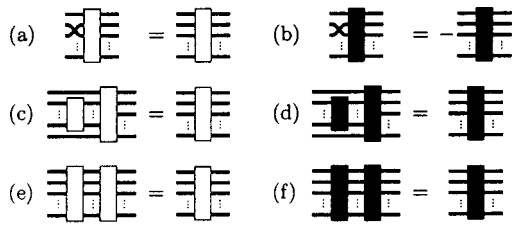


FIG. 1. Properties of the diagrammatic symmetrization and antisymmetrization operators.

where the index b can be omitted, as indeed can all other "dummy" indices. The Kronecker delta is drawn as

$$\delta_b^a = b \longleftarrow a \quad a,b = 1,2,\ldots,n,$$

and its trace—the dimension of the representation—is drawn as a closed loop,

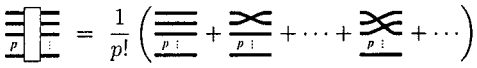
$$\bigcirc = \delta_a^a = n. \tag{2}$$

Index permutations can be drawn in terms of Kronecker deltas. For example, the symmetric group S_2 acting on two indices consists of the identity element $\mathbb{I}_{ab}{}^{cd} = \delta_a^d \delta_b^c$ and the transposition $(12)_{ab}{}^{cd} = \delta_a^c \delta_b^d$. In the diagrammatic notation these operators are drawn as

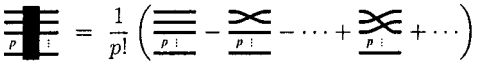
$$\mathbb{I}_{ab}{}^{cd} = \begin{matrix} a \longleftarrow d \\ b \longleftarrow c \end{matrix} \quad \text{and} \quad (12)_{ab}{}^{cd} = \begin{matrix} a \\ b \end{matrix} \times \begin{matrix} d \\ c \end{matrix}.$$

Symmetrization of p indices is achieved by adding all permutations σ of p indices, $S = (1/p!)\Sigma_{\sigma \in S_p} \delta_{\sigma(b1)}^{a_1} \ldots \delta_{b_p)}^{a_p}$. Similarly, the operator $A = (1/p!)\Sigma_{\sigma \in S_p} \mathrm{sgn}(\sigma)\, \delta_{\sigma(b_1}^{a_p} \ldots \delta_{b_p)}^{a_1}$ (with a minus − for odd permutations) antisymmetrizes p indices. Combinations of symmetrizers S and antisymmetrizers A are collectively referred to as *symmetry operators*.

In the diagrammatic notation we write the symmetrizers and the antisymmetrizers of *length p* as[19]



(3)



(4)

In order to streamline the notation we shall neglect the arrows whenever this leads to no confusion. Basic properties of the symmetry operators are listed in Fig. 1: A symmetrizer is invariant under any permutation of its legs, rule (a). The antisymmetrizer changes sign under odd permutations, rule (b). A symmetrizer connected by more than one line to an antisymmetrizer is zero by rules (a) and (b),

(5)

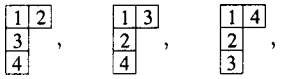
Recursive identities for the (anti)symmetrizers are given in (A5) and (A4).

III. YOUNG TABLEAUX

Partition k identical boxes into D subsets, and let λ_m, $m=1,2,\ldots,D$, be the number of boxes in the subsets ordered so that $\lambda_1 \geqslant \lambda_2 \cdots \geqslant \lambda_D \geqslant 1$. Then the partition $\lambda=[\lambda_1,\lambda_2,\cdots,\lambda_D]$ fulfills $\Sigma_{m=1}^{D}\lambda_m=k$. The diagram obtained by drawing the D rows of boxes on top of each other, left aligned, starting with λ_1 at the top, is called a *Young diagram* Y.

Inserting each number from the set $\{1,\ldots,k\}$ into a box of a Young diagram Y in such a way that numbers increase when reading a column from top to bottom and numbers do not decrease when reading a row from left to right yields a *Young tableau* Y_a. The subscript a labels different tableaux derived from a given Young diagram, i.e., different admissible ways of inserting the numbers into the boxes. A *standard tableau* is a k-box Young tableau constructed by inserting the numbers $1,\ldots,k$ according to the above rules, but using each number exactly once.

As an example, three distinct standard tableaux,



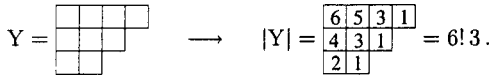
are obtained from the four-box Young diagram with partition $\lambda=[2,1,1]$.

A. Symmetric group S_k

Young diagrams label the irreps of the symmetric group S_k. A k-box Young diagram Y corresponds to an irrep of S_k, and Δ_λ, the dimension of the irrep λ, is the number of standard tableaux Y_a that can be constructed from the Young diagram Y. From the above example we see that the irrep $\lambda=[2,1,1]$ of S_4 is three dimensional. The formula for the dimension Δ_Y of the irrep of S_k corresponding to the Young diagram Y is

$$\Delta_Y = \frac{k!}{|Y|}. \tag{6}$$

The number $|Y|$ is computed using a "hook" rule: Enter into each box of the Young diagram the number of boxes below and to the left of the box, including the box itself. Then $|Y|$ is the product of the numbers in all the boxes. For instance,



The hook rule (6) was first proved surprisingly late, in 1954, by Frame, de B. Robinson, and Thrall.[27] Various proofs can be found in Refs.[15,16,28–31]; in particular, see Sagan[32] and references therein.

B. Representations of U(n)

While every Young diagram labels an irrep of S_k, every standard tableau labels an irrep of U(n). The dimension d_Y of an irrep labeled by the Young diagram Y equals the number of Young tableaux Y_a that can be obtained from Y by inserting numbers from the set $\{1,2,\ldots,n\}$ such that the numbers increase in each column and do not decrease in each row.

For example, for SU (2) the partition [2] corresponds to a three-dimensional irrep with tableaux $\boxed{1}\boxed{1}$, $\boxed{1}\boxed{2}$, and $\boxed{2}\boxed{2}$, and the partition [1, 1] corresponds to a one-dimensional irrep with one tableau, $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$. Similarly, one can check that for SU (3), the partition [2] is six dimensional and the partition [1, 1] is three dimensional. We shall derive the dimension formula for any irrep of U(n) in Sec. IV C.

IV. YOUNG PROJECTION OPERATORS

We now present a diagrammatic method for construction of Young projection operators. A combinatorial version of these operators was given by van der Waerden,[33] who credited von Neumann. There are many other versions in the literature, all of them illustrating the fundamental theorem of 't Hooft and Veltman:[34] combinatorics cannot be taught. What follows might aid those who think visually.

A. The group algebra

Our goal is to construct the projection operators such as (1) for any irrep of S_k. We need to construct a basis set of invariant tensors, multiply them by scalars, add and subtract them, and multiply a tensor by another tensor. The necessary framework is provided by the notion of *group algebra*.

The elements $\sigma \in S_k$ of the symmetric group S_k form a basis of a $k!$-dimensional vector space V of elements

$$s = \sum_{\sigma \in S_k} s_\sigma \sigma \in V, \tag{7}$$

where s_σ are the components of the vector s in the given basis. If s, $t \in V$ have components (s_σ) and (t_σ), we define the product of s and t as the vector st in V with components $(st)_\sigma = \Sigma_{\tau \in S_k} s_\tau t_{\tau^{-1}\sigma}$. This multiplication is associative because it relies on the associative group operation. Since V is closed under the multiplication the elements of V form an associative algebra—the *group algebra* of S_k. Acting on an element $s \in V$ with any group element maps s to another element in the algebra, hence this map gives a $k!$-dimensional matrix representation of the group algebra, the *regular representation*. Note that the matrices of any representation μ of the group is also a basis for representation of the algebra: Let $D^\mu(\sigma)$ denote a (possibly reducible) representation of S_k. The group algebra of S_k in the representation μ then consists of elements

$$D^\mu(\mathrm{s}) = \sum_{\sigma \in S_k} \mathrm{s}_\sigma D^\mu(\sigma) \in V,$$

where s is given by (7). The minimal left-ideals V_λ of the group algebra (i.e., $sV_\lambda = V_\lambda$ for all $s \in V$, and V_λ has no proper subideals) are the proper invariant subspaces corresponding to the irreps of the symmetric group S_k.

The regular representation is reducible and each irrep appears Δ_λ times in the reduction, where Δ_λ is the dimension of the subspace V_λ corresponding to the irrep λ. This gives the well-known relation between the order of the symmetric group $|S_k| = k!$ (the dimension of the regular representation) and the dimensions of the irreps,

$$|S_k| = \sum_{\text{irreps } \lambda} \Delta_\lambda^2.$$

Using (6) and the fact that the Young diagrams label the irreps of S_k, we have

$$1 = k! \sum_{(k)} \frac{1}{|\mathrm{Y}|^2}, \tag{8}$$

where the sum is over all Young diagrams with k boxes. We shall use this relation to determine the normalization of Young projection operators in the Appendix.

The reduction of the regular representation of S_k gives a completeness relation

$$\mathbb{I} = \sum_{(k)} P_{\mathrm{Y}}$$

into projection operators

$$P_{\mathrm{Y}} = \sum_{\mathrm{Y}_a \in \mathrm{Y}} P_{\mathrm{Y}_a}.$$

The sum is over all Young tableaux derived from the Young diagram Y. Each P_{Y_a}. projects onto the corresponding invariant subspace V_{Y_a}. For each Y there are Δ_{Y} such projection operators (corresponding to the Δ_{Y} possible standard arrangements of the diagram) and each of these project onto one of the Δ_{Y} invariant subspaces V_{Y} of the reduction of the regular representation. It follows that the projection operators are orthogonal and that they constitute a complete set.

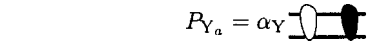
B. Diagrammatic Young projection operators

We now generalize (1), the S_2 projection operators expressed in terms of Kronecker deltas, to Young projection operator for any S_k.

The Kronecker delta is invariant under unitary transformations, $\delta^b_a = (U^\dagger)_a{}^{a'} \delta^{b'}_{a'} U_{b'}{}^b$, $U \in \mathrm{U}(n)$, and so is any combination of Kronecker deltas, such as the symmetrizers of Fig. 1. Since these operators constitute a complete set, any U(n) invariant tensor built from Kronecker deltas can be written in terms of symmetrizers and antisymmetrizers. In particular, the invariance of the Kronecker delta under U(n) transformations implies that the same symmetry group operators which project the irreps of S_k also yield the irreps of U(n).

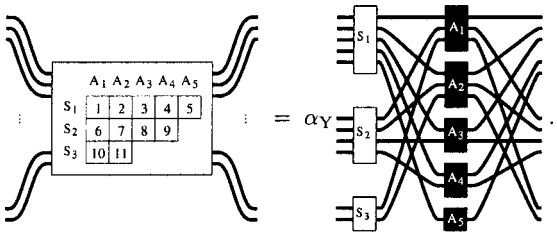
The simplest examples of Young projection operators are those associated with the Young tableaux consisting of either one row or one column. The corresponding Young projection operators are simply the symmetrizers (3) or the antisymmetrizers (4), respectively. As projection operators for S_k, the symmetrizer projects onto the one-dimensional subspace corresponding to the fully symmetric representation, and the antisymmetrizer projects onto the alternating representation.

A Young projection operator for a mixed symmetry Young tableau will here be constructed by first antisymmetrizing subsets of indices, and then symmetrizing other subsets of indices; which subsets are dictated by the form of the Young tableau, as will be explained shortly. Schematically,

$P_{\mathrm{Y}_a} = \alpha_{\mathrm{Y}}$  ,

where α_{Y} is a normalization constant (defined below) ensuring that the operators are idempotent, $P_{\mathrm{Y}_a} P_{\mathrm{Y}_b} = \delta_{ab} P_{\mathrm{Y}_a}$. This particular form of projection operators is by no means unique—Young projection operators symmetric under transposition are constructed in Ref. 18—but is particularly convenient for explicit computations

Let Y_a be a k-box standard tableau. Arrange a set of symmetrizers corresponding to the rows in Y_a, and to the right of this arrange a set of antisymmetrizers corresponding to the columns in Y_a. For a Young diagram Y with s rows and t columns we label the rows $\mathrm{S}_1, \mathrm{S}_2, \ldots, \mathrm{S}_s$ and to the columns $\mathrm{A}_1, \mathrm{A}_2, \ldots, \mathrm{A}_t$. Each symmetry operator in P_{Y_a} is associated with a row/column in Y_a, hence we label a symmetry operator after the corresponding row/column, for example



Let the lines numbered 1 to k enter the symmetrizers as described by the numbers in the boxes in the standard tableau and connect the set of symmetrizers to the set of antisymmetrizers in a nonvanishing way, avoiding multiple intermediate lines prohibited by (5). Finally, arrange the lines coming out of the antisymmetrizers such that if the lines all passed straight through the symmetry operators, they would exit in the same order as they entered.

We shall denote by Δ_Y the dimensions of irreps of S_k, and by d_Y the dimensions of irreps of U(n). Let $|S_i|$ or $|A_i|$ denote the number of boxes within a row or column, respectively. Thus $|A_i|$ also denotes the number of lines entering the antisymmetrizer A_i, and similarly for the symmetrizers. The normalization constant α_Y is given by

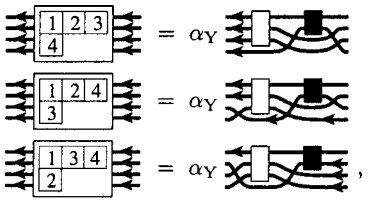
$$\alpha_Y = \frac{\Pi_{i=1}^{s}|S_i|!\Pi_{j=1}^{t}|A_j|!}{|Y|},$$

where $|Y|$ is related through (6) to Δ_Y, the dimension of irrep Y of S_k, and is a hook rule S_k combinatoric number. The normalization depends only on the shape of the Young diagram, not the particular tableau. The Young projection operators

(1) are *idempotent*, $P_Y^2 = P_Y$;
(2) are *orthogonal*: if Y and Z are two distinct standard tableaux, then $P_Y P_Z = P_Z P_Y = 0$; and
(3) constitute a *complete set*, $\mathbb{I} = \Sigma P_Y$, where the sum is over all standard tableaux Ywith k boxes.

The projections are unique up to an overall sign. By construction, the identity element always appears as a term in the expansion of the symmetry operators of the Young projection operators—the overall sign is fixed by requiring that the identity element comes with a positive coefficient. The diagrammatic proof that the above rules indeed assign a unique projection operator to each standard tableaux is the central result of this paper; as it would impede the flow of our argument at this point, it is placed in the Appendix.

Example: The Young diagram corresponding to the partition $[3,1]$ tells us to use one symmetrizer of length three, one of length one, one antisymmetrizer of length two, and two of length one. There are three distinct standard tableaux, each corresponding to a projection operator

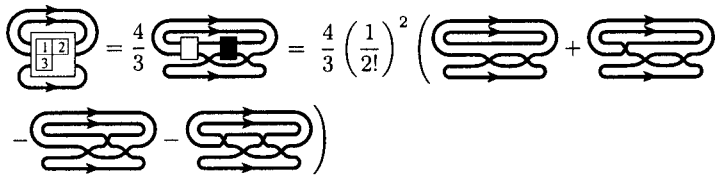


where α_Y is the normalization constant. The symmetry operators of unit width need not be drawn explicitly. We have $|Y| = 8$, $|S_1| = 3$, $|S_2| = 1$, $|A_1| = 2$, etc., yielding the normalization $\alpha_Y = 3/2$.

C. Dimensions of U(n) irreps

The dimension d_Y of a U(n) irrep is computed by taking the trace of the corresponding Young projection operator, $d_Y = \mathrm{tr}\, P_Y$. The trace can be evaluated by expanding the symmetry operators using (3) and (4). By (2), each closed line is worth n, so d_Y is a polynomial in n of degree k.

Example: The dimension of a three-index Young projection operator,

d_Y =  (9)

$$= \frac{1}{3}(n^3 + n^2 - n^2 - n) = \frac{n(n^2-1)}{3}. \tag{10}$$

Such brute expansion is unnecessarily laborious: The dimension of the irrep labeled by Y is

$$d_Y = \frac{f_Y(n)}{|Y|}, \tag{11}$$

where $f_Y(n)$ is the polynomial in n obtained from the Young diagram Y by multiplying the numbers written in the boxes of Y, according to the following rules: (A) The upper left box contains an n. (B) The numbers in a row increase by 1 when reading from left to right. (C) The numbers in a column decrease by 1 when reading from top to bottom. Hence, if k is the number of boxes in Y, $f_Y(n)$ is a polynomial in n of degree k. The dimension formula (11) is well known, see for instance Ref. 11.

In the example (10), we have $f_Y(n)=n(n-1)(n+1)$ and $|Y|=3$, giving $d_Y=n(n^2-1)/3$.

Example: For $Y=[4,2,1]$ we have

$$d_Y = \begin{array}{|c|c|c|c|}\hline n & n+1 & n+2 & n+3 \\ \hline n-1 & n \\ \cline{1-2} n-2 \\ \cline{1-1} \hline\end{array} \Big/ \begin{array}{|c|c|c|c|}\hline 6 & 4 & 2 & 1 \\ \hline 3 & 1 \\ \cline{1-2} 1 \\ \cline{1-1} \hline\end{array} = \frac{n^2(n^2-1)(n^2-4)(n+3)}{144}.$$

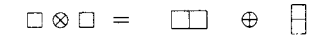
A diagrammatic proof of the U(n) dimension formula (11) is given in the Appendix.

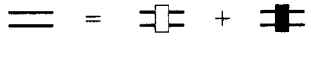
Diagrammatically, the number $f_Y(n)$ is the number of n-color colorings of the strand network corresponding to tr P_Y, see, for example, Ref. 18.

D. Examples

We present examples to illustrate decomposition of reducible representation into irreps using the diagrammatic projection operators.

The Young diagram □ corresponds to the fundamental n-dimensional irrep of U(n). As we saw in (1), the direct product of two of these n-dimensional representations is a n^2-dimensional reducible representation,

$\square \otimes \square$ =  (12)

 (13)

$$n^2 = \frac{n(n+1)}{2} + \frac{n(n-1)}{2}. \tag{14}$$

Equation (12) shows the decomposition of the reducible representation in terms of Young diagrams, and (13) gives the corresponding projection operators. Tracing (13) yields the dimensions (14) of the irreps.

The first nontrivial example is the reduction of the three-index tensor Young projection operators, listed in Fig. 2. Further examples can be found in Ref. 18.

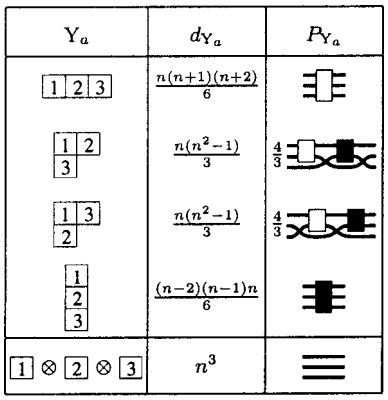
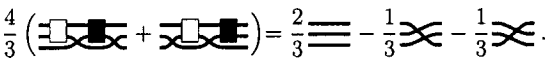


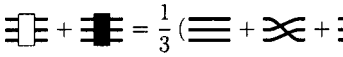
FIG. 2. Reduction of a three-index tensor. Bottom row; the direct product of three unit tableaux, the sum of dimensions, and the projection operators completeness sum.

The four projectors are orthogonal by inspection. In order to verify the completeness, expand first the two three-index projection operators of mixed symmetry,



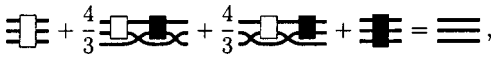
(15)

In the sum of the fully symmetric and the fully antisymmetric tensors all the odd permutations cancel, and we are left with



(16)

Adding (15) and (16) we find



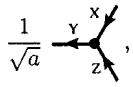
verifying the completeness relation.

Acting with any permutation on the fully symmetric or antisymmetric projection operators gives ± 1 times the projection operator (see Fig. 1). For projection operators of mixed symmetry the action of a permutation is not as simple, because the permutations will mix the spaces corresponding to the different tableaux. Here we shall need only the action of a permutation within a $3n-j$ coefficient, and, as we shall show below, in this case the result will again be simple, a factor ± 1 or 0.

V. RECOUPLING RELATIONS

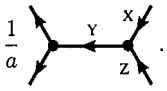
In the spirit of Feynman diagrams, group theoretic weights with all indices contracted can be drawn as "vacuum bubbles." We now show that for U(n) any such vacuum bubble can be evaluated diagrammatically, either directly, as a $3n-j$ coefficient, or following a reduction to $3-j$ and $6-j$ coefficients. The exposition of this section follows closely Ref. 18; the reader can find there more details, as well as the precise relationship between our $3-j$ and $6-j$ coefficients, and the Wigner $3-j$ and $6-j$ symbols.[35]

The decomposition of a many-particle state can be implemented sequentially, decomposing two-particle states at each step. The Clebsch–Gordan coefficients for $X \otimes Z \rightarrow Y$ can be drawn as *3-vertices*,

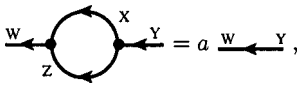


(17)

where $1/\sqrt{a}$ is an (arbitrary) normalization constant. The projection operators for $X \otimes Z \to Y \to X \otimes Z$ can be drawn as

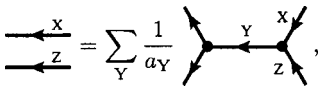


The orthogonality of irreps implies $W = Y$ in



(18)

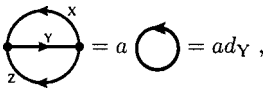
and the completeness relation can be drawn as



(19)

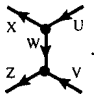
where the sum is over all irreps contained in $X \otimes Z$.

The normalization constant a can be computed by tracing (18),

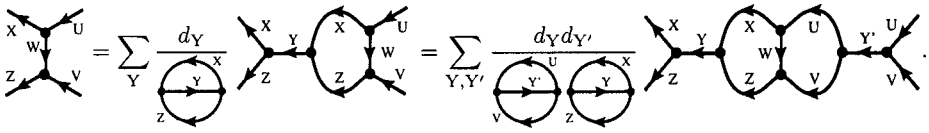


where d_Y is the dimension of the representation Y. The vacuum bubble on the left-hand side is called a $3-j$ coefficient. More generally, vacuum bubbles with n lines are called $n-j$ coefficients.

Let particles in representations U and V interact by exchanging a particle in the representation W, with the final state particles in the representations X and Z,



Applying the completeness relation (19) repeatedly yields



By the orthogonality of irreps $Y = Y'$, and we obtain the *recoupling relation*

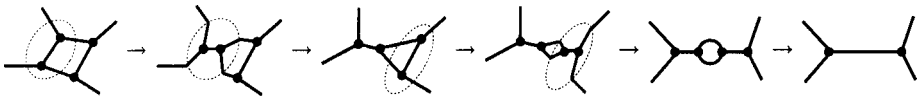
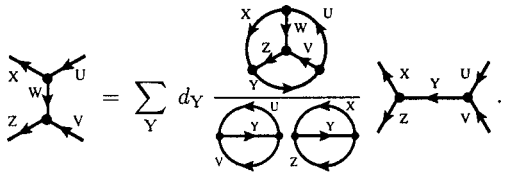


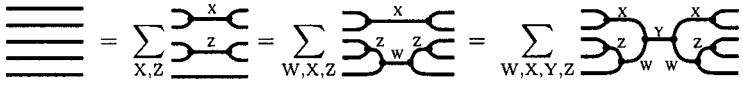
FIG. 3. A reduction of a 4-vertex loop to a sum of "tree" tensors, weighted by products of $3-j$ and $6-j$ coefficients.



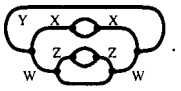
(20)

The "Mercedes" vacuum bubbles in the numerators are called $6-j$ coefficients. *Any* arbitrarily complicated vacuum bubble can be reduced to $3-j$ and $6-j$ coefficients by recursive use of the recoupling relation (20). For instance, a four-vertex loop can be reduced to a two-vertex loop by repeated application of the recoupling relations as sketched in Fig. 3.

Another, more explicit example of a sequence of recouplings, is the following step-by-step reduction of a five-particle state:



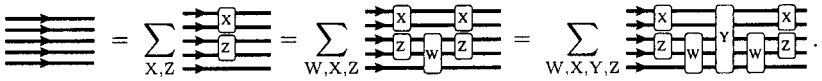
(for brevity we omit the normalization factors here). Taking the trace of both sides leads to $12-j$ coefficients of the form



(21)

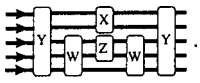
A. U(n) recoupling relations

Due to the overall particle number conservation (we consider no "antiparticle" states here), for U(n) the above five-particle recoupling flow takes a very specific form in terms of Young projection operators,



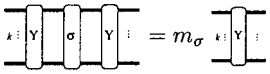
More generally, we can visualize any sequence of U(n) pairwise Clebsch–Gordan reductions as a flow with lines joining into thicker and thicker projection operators, always ending in a maximal P_Y which spans across all lines.

In the trace (21) we can use the idempotency of the projection operators to double the maximal Young projection operator P_Y, and sandwich by it all smaller projection operators,



(22)

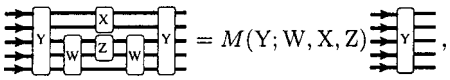
From uniqueness of the connection between the symmetry operators (see the Appendix I) we have for any permutation $\sigma \in S_k$



(23)

where $m_\sigma = 0, \pm 1$. Expressions like (22) can be evaluated by expanding the projection operators

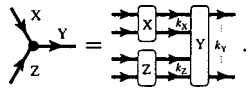
P_W, P_X, P_Z and determining the value of m_σ of (23) for each permutation σ of the expansion. The result is



where the factor $M(\mathrm{Y};\mathrm{W},\mathrm{X},\mathrm{Z})$ *does not depend* on n and is determined by a purely symmetric group calculation. Several examples follow.

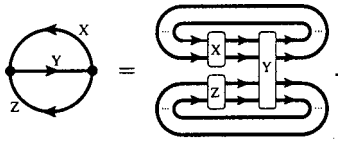
B. Evaluation of 3*n*-*j* coefficients

Let X, Y, and Z be irreps of U(n). In terms of the Young projection operators P_X, P_Y, and P_Z, a U(n) three-vertex (17) is obtained by tying together the three Young projection operators,



(24)

The number of particles is conserved (the multiparticle states constructed here consist only of particles, no "antiparticles"): $k_X+k_Z=k_Y$. A $3-j$ coefficient constructed from the vertex (24) is then

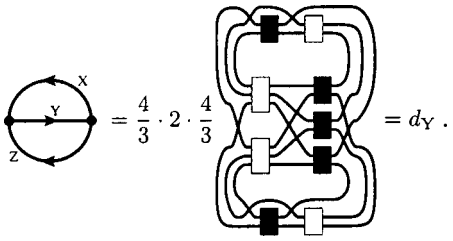


(25)

As an example, take

$$\mathrm{X}=\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\,,\quad \mathrm{Y}=\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\cline{1-3}\end{array}\,,\qquad \text{and}\qquad \mathrm{Z}=\begin{array}{|c|c|}\hline 4&5\\\hline 6\\\cline{1-1}\end{array}\,.$$

Then

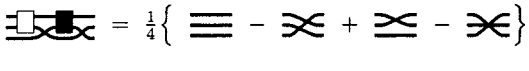


(26)

In principle the value of such $3-j$ coefficient can be computed by expanding out all symmetry operators, but that is not recommended as the number of terms in such expansions grows combinatorially with the total number of boxes in the Young diagram Y. Instead, the answer—in this case $d_Y=(n^2-1)n^2(n+1)(n+2)/144$— is obtained as follows.

In general, the $3-j$ coefficients (25) can be evaluated by expanding the projections P_X and P_Z and determing the value of m_σ in (23) for each permutation σ of the expansion.

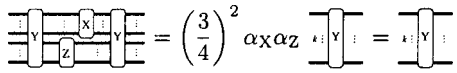
As an example, consider the $3-j$ coefficient (26). With P_Y as in (26) we find



$$m_\sigma(P_X): \qquad +1 \qquad 0 \qquad +1 \qquad -1$$

$$m_\sigma(P_Z): \qquad +1 \qquad -1 \qquad 0 \qquad -1\,,$$

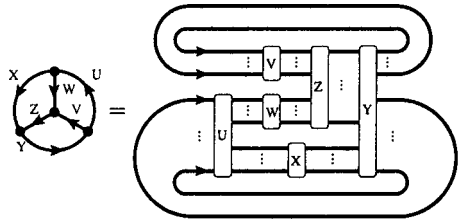
hence



and the value of the $3-j$ is d_{Y} as claimed in (26). That the eigenvalue happens to be 1 is an accident—in tabulations of $3-j$ coefficients[26] it takes a range of values.

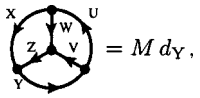
The relation (23) implies that the value of any U(n) $3-j$ coefficient (25) is $M(\mathrm{Y};\mathrm{X},\mathrm{Z})d_{\mathrm{Y}}$, where d_{Y} is the dimension of the maximal irrep Y.

A $6-j$ coefficient is composed of the three-vertex (17) and the other three-vertex in the projection operator (24), with all arrows reversed. A general U(n) $6-j$ coefficient has the form



(27)

Using the relation (23) we immediately see that



(28)

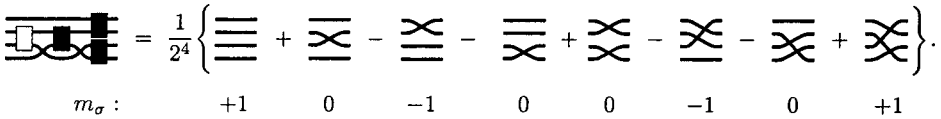
where M is a pure symmetric group $S_{k\mathrm{Y}}$ number, independent of U(n); it is surprising that the only vestige of U(n) is the fact that the value of a $6-j$ coefficient is proportional to the dimension d_{Y} of its largest projection operator.

Example: Consider the $6-j$ constructed from the Young tableaux,

$$\mathrm{U}=\begin{array}{|c|c|}\hline 2&3\\\hline 4\\\cline{1-1}\end{array},\quad \mathrm{V}=\begin{array}{|c|}\hline 1\\\cline{1-1}\end{array},\quad \mathrm{W}=\begin{array}{|c|}\hline 2\\\cline{1-1}\end{array},$$

$$\mathrm{X}=\begin{array}{|c|}\hline 3\\\hline 4\\\cline{1-1}\end{array},\quad \mathrm{Y}=\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\hline 4\\\cline{1-1}\end{array},\quad \mathrm{Z}=\begin{array}{|c|}\hline 1\\\hline 2\\\cline{1-1}\end{array}.$$

Using the idempotency we can double the projection P_{Y} and sandwich the other operators, as in (22). Several terms cancel in the expansion of the sandwiched operator, and we left with

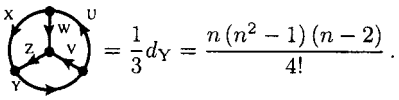


m_σ : +1 0 −1 0 0 −1 0 +1

We have listed the symmetry factors m_σ of (23) for each of the permutations σ sandwiched between the projection operators P_{Y}. We find that in this example the symmetric group factor M of (28) is

$$M=\frac{4}{2^4}\alpha_{\mathrm{U}}\alpha_{\mathrm{V}}\alpha_{\mathrm{W}}\alpha_{\mathrm{X}}\alpha_{\mathrm{Z}}=\frac{1}{3},$$

so the value of the $6-j$ is



The method generalizes to evaluations of any $3n-j$ coefficients of U(n).

C. Sum rules

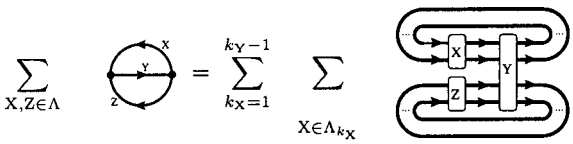
Let Y be a standard tableau with k_Y boxes, and let Λ be the set of all standard tableaux with one or more boxes (exclude the trivial $k=0$ representation). Then the $3-j$ coefficients obey the sum rule

$$\sum_{X,Z\in\Lambda} \quad = \quad (k_Y-1)d_Y. \tag{29}$$

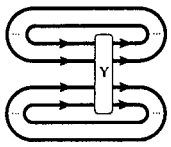
The sum is finite, because the $3-j$ is nonvanishing only if the number of boxes in X and Z add up to k_Y, and this happens only for a finite number of tableaux.

To prove (29), recall that the Young projection operators constitute a complete set, $\Sigma_{X\in\Lambda_k}P_X=\mathbb{I}$, where $\mathbb{I}$ is the $k\times k$ unit matrix and Λ_k the set of all standard tableaux of Young diagrams with k boxes. Hence

$$\sum_{X,Z\in\Lambda} \quad = \sum_{k_X=1}^{k_Y-1} \sum_{\substack{X\in\Lambda_{k_X}\\ Z\in\Lambda_{k_Y-k_X}}}$$



$$= \sum_{k_X=1}^{k_Y-1}$$



$$= \sum_{k_X=1}^{k_Y-1} d_Y \;=\; (k_Y-1)d_Y\,.$$

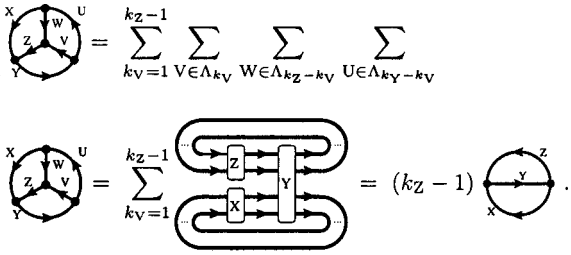
This sum rule offers a useful cross-check on tabulations of $3-j$ values, see for instance Ref. 26.

There is a similar sum rule for the $6-j$ coefficients,

$$\sum_{X,Z,U,V,W\in\Lambda} \quad = \quad \frac{1}{2}(k_Y-1)(k_Y-2)\,d_Y\,. \tag{30}$$

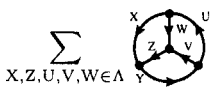
Referring to the $6-j$ (27), let k_U be the number of boxes in the Young diagram U, k_X be the number of boxes in X, etc., and let k_Y be given. From (27) we see that k_X takes values between 1 and k_Y-2 and k_Z takes values between 2 and k_Y-1, subject to the constraint $k_X+k_Z=k_Y$. We now sum over all tableaux U, V, and W keeping k_Y, k_X, and k_Z fixed. Note that k_V can take values 1,…, k_Z-1. Using completeness we find

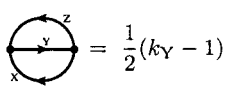
$$\sum_{U,V,W\in\Lambda} = \sum_{k_V=1}^{k_Z-1} \sum_{V\in\Lambda_{k_V}} \sum_{W\in\Lambda_{k_Z-k_V}} \sum_{U\in\Lambda_{k_Y-k_V}}$$



Now sum over all tableaux X and Z to find

$$\sum_{X,Z,U,V,W\in\Lambda} \quad = \sum_{k_Z=2}^{k_Y-1}(k_Z-1) \sum_{Z\in\Lambda_{k_Z}} \sum_{X\in\Lambda_{k_Y-k_Z}} \quad = \frac{1}{2}(k_Y-1)(k_Y-2)\,d_Y$$





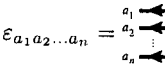
verifying the sum rule (30) for $6-j$ coefficients.

VI. SU(N) AND ITS ADJOINT REPRESENTATION

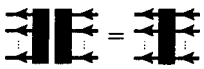
The SU(n) group elements satisfy det U=1, so SU(n) has an additional invariant, the Levi–Civita tensor

$$\epsilon_{a_1 a_2 \cdots an} = U_{a_1}{}^{a_1'} U_{a_2}{}^{a_2'} \cdots U_{a_n}{}^{a_n'} \epsilon_{a_1' a_2' \cdots a_n'}.$$

In the diagrammatic notation the Levi–Civita tensors can be drawn as[20]

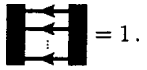
$$\frac{1}{\sqrt{n!}} \varepsilon_{a_1 a_2 \ldots a_n} =$$  , $$\frac{1}{\sqrt{n!}} \varepsilon^{a_n \ldots a_2 a_1} =$$  .

They satisfy

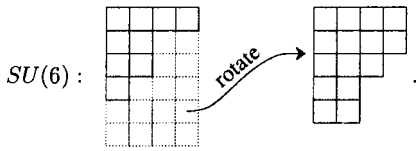


(31)

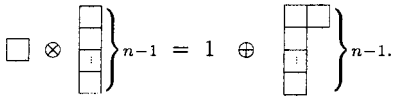
(Levi–Civita projects an n-particle state onto a single, one-dimensional, singlet representation), and are correctly normalized,



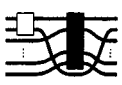
The Young diagrams for SU(n) cannot contain more than n rows, since at most n indices can be antisymmetrized. By contraction with the Levi-Civita tensor, a column with k boxes can be converted into a column of $n-k$ boxes: this operation associates to each irrep the *conjugate* irrep. The Young diagram corresponding to the irrep is the *conjugate* Young diagram constructed from the missing pieces needed to complete the rectangle of n rows. For example, the conjugate of the irrep corresponding to the partition [4, 2, 2, 1] of SU(6) has the partition [4, 4, 3, 2, 2]:



The Levi–Civita tensor converts an antisymmetrized collection of $n-1$ "in" indices, an ($n-1$)-particle state, into 1 "out" index: a single antiparticle state $\overline{\square}$, the conjugate of the fundamental representation $\square$ single particle state. The corresponding Young diagram is a single column of $n-1$ boxes. The product of the fundamental representation and the conjugate representation of SU(n) decomposes into a singlet and the adjoint representation,



In the notation introduced in Sec. IV, the Young projection operator for the adjoint representation A is drawn as

$$P_A = \frac{2(n-1)}{n}$$ 

Using P_A and the definition (24) of the three-vertex, SU(n) group theory weights involving quarks, antiquarks, and gluons can be calculated by expansion of the symmetry operators or by application of the recoupling relation. When the adjoint representation plays a key role, as it does in gauge theories, it is wisest to abandon the above construction of all irreps by Clebsch–Gordan reductions of multiparticle states, and build the theory by taking a single particle and a single antiparticle as the fundamental building blocks. A much richer theory, beyond the scope of this paper follows,

leading to a diagrammatic construction of representations of all simple Lie groups, the classical as well as the exceptional. The reader is referred to Ref. 18 for the full exposition.

VII. NEGATIVE DIMENSIONS

We conclude by a brief discussion of implications of the $n \to -n$ duality[18,36] of U(n) invariant scalars.

Any SU(n) invariant tensor is built from Kronecker deltas and Levi–Civita tensors. A scalar is a tensor object with all indices contracted, so in the diagrammatic notation a scalar is a diagram with no external legs, a vacuum bubble. Thus, in scalars Levi–Civita tensors can appear only in pairs (the lines must end somewhere), and by (31) the Levi–Civita tensors combine to antisymmetrizers. Consequently both U(n) and SU(n) invariant scalars are all built only from symmetrizers and antisymmetrizers.

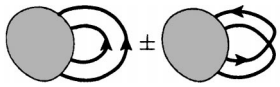
Expanding all symmetry operators in a U(n) vacuum bubble gives a sum of entangled loops. Each loop is worth n, so each term in the sum is a power of n, and therefore a U(n) invariant scalar is a polynomial in n.

The negative dimensionality theorem[18,36] for U(n) states that interchanging symmetrizers and antisymmetrizers in U(n) invariant scalar is equivalent (up to an overall sign) to substituting $n \to -n$ in the polynomial, which is the value of the scalar. We write this

$$\overline{\mathrm{U}(n)} = \mathrm{U}(-n).$$

The bar symbolizes the interchange of symmetrizers and antisymmetrizers.

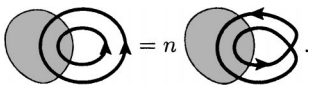
The terms in the expansion of all symmetry operators in a U(n) vacuum bubble can be arranged in pairs that only differ by one crossing,



(32)

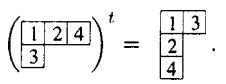
with $\pm$ depending on whether the crossing is due to symmetrization $(+)$ or antisymmetrization $(-)$. The gray blobs symbolize the tangle of lines common to the two terms.

If the two arcs outside the gray blob of the first term of (32) belong to separate loops, then in the second term they will belong to the same loop. The two terms thus differ only by a factor of n: schematically,



Likewise, if the arcs in the first term belong to the same loop then in the second term they will belong to two separate loops. In this case the first term is $1/n$ times the second term. In either case the ratio of the two terms is an odd power of n. Interchanging symmetrizers and antisymmetrizers in a U(n) vacuum bubble changes the sign in (32). Up to an overall sign the result is the same as substituting $n \to -n$. This proves the theorem.

Consider now the implications for the dimension formulas and the values of $3n-j$ coefficients. The dimension of an irrep of U(n) is the trace of the Young projection operator, a vacuum bubble diagram built from symmetrizers and antisymmetrizers. Applying the negative dimensionality theorem we get $d_{Y^t}(n) = d_Y(-n)$, where Y^t is the *transpose* Y^t of the standard Young tableau Y obtained by interchanging rows and columns (reflection across the diagonal). For instance [3, 1] is the transpose of [2, 1, 1],



As an example, note the $n \to -n$ dualities in the dimension formulas of Fig. 2.

Now for standard tableaux X, Y, and Z, compare the diagram of the $3-j$ constructed from X, Y, and Z to that constructed from X^t, Z^t, and Y^t. The diagrams are related by a reflection in a vertical line, reversal of all the arrows on the lines, and interchange of symmetrizers and antisymmetrizers. The first two operations do not change the value of the diagram, hence the values of the two diagrams are again related by $n \leftrightarrow -n$ (and possibly an overall sign; this sign is fixed by requiring that the highest power of n comes with a positive coefficient). Hence in tabulation it is sufficient to calculate approximately half of all $3-j$'s. The $3-j$ sum rule (29) provides a cross-check.

The two $6-j$ coefficients

are related by a reflection in a vertical line, reversal of all the arrows on the lines, and interchange of symmetrizers and antisymmetrizers—this can be seen by writing out the $6-j$ coefficients in terms of the Young projection operators as in (27). By the negative dimensionality theorem, the values of the two $6-j$ coefficients are therefore again related by $n \leftrightarrow -n$.

VIII. SUMMARY

We have presented a diagrammatic method for construction of correctly normalized Young projection operators for U(n). These projection operators in diagrammatic form are useful for explicit evaluation of group theoretic quantities such as the $3n-j$ coefficients. Using the recoupling relations, all U(n) invariant scalars can be reduced to expressions involving only terms of $3-j$ and $6-j$ coefficients and the dimensionalities of the representations. Our main results are as follows:

(i) Diagrammatic Young projection operators for tensors (multiparticle states) with given symmetry properties; a diagrammatic proof of their uniqueness, completeness, and orthogonality.
(ii) U(n) invariant scalars may be expressed in terms of the Young projection operators, and their values computed by diagrammatic expansions.
(iii) U(n) $3-j$ and $6-j$ coefficients constructed from the three-vertex defined in (24) have simple n-dependencies: they are proportional to the dimension of the maximal irrep projection operator that spans over all multiparticle indices.
(iv) The negative dimensionality theorem applies to all U(n) invariant scalars, in particular the $3n-j$ coefficients and the dimensions of the irreps of U(n).
(v) The sum rules (29) and (30) for $3-j$ and $6-j$ coefficients afford useful cross-checks of $3n-j$ tabulations.

ACKNOWLEDGMENTS

One of the authors (H.E.) would like to thank the University of Edinburgh, UK, the Niels Bohr Institute, Copenhagen, and the Center for Nonlinear Studies, Georgia Tech, for hospitality, and the Danish Research Council for support. We thank M. Hermele for bringing Ref. 32 to our attention.

APPENDIX: DIAGRAMMATIC YOUNG PROJECTION OPERATORS: THE PROOFS

In this appendix we prove the properties of the Young projection operators stated above in Sec. IV.

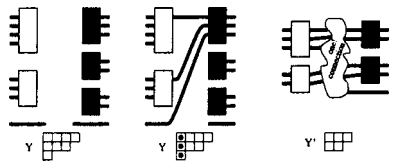


FIG. 4. There is a unique (up to an overall sign) connection between the symmetrizers and the antisymmetrizers, so the Young projection operators are well defined by the construction procedure explained in the text. The figure shows the principles of the proof. The dots on the middle Young diagram mark boxes that correspond to contracted lines.

Uniqueness

We show that the Young projection operators P_{Y} are well defined by proving the existence and uniqueness (up to an overall sign) of a nonvanishing connection between the symmetrizers and antisymmetrizers in P_{Y}.

The proof is by induction over the number of columns t in the Young diagram Y; the principles are illustrated in Fig. 4. For $t=1$ the Young projection operator consists of one antisymmetrizer of length s and s symmetrizers of length 1, and clearly the connection can only be made in one way, up to an overall sign, see Fig. 1(b).

Assume the result to hold for Young projection operators derived from Young diagrams with $t-1$ columns. Let Y be a Young diagram with t columns. The lines from A_1 in P_{Y} must connect to different symmetrizers for the connection to be nonzero. Since there are exactly $|\mathrm{A}_1|$ symmetrizers in P_{Y}, this can be done in essentially one way, since which line goes to which symmetrizer is only a matter of an overall sign, and where a line enters a symmetrizer is irrelevant due to Fig. 1(a).

After having connected A_1, connecting the symmetry operators in the rest of P_{Y} is the problem of connecting symmetrizers to antisymmetrizers in the Young projection operator $P_{\mathrm{Y}'}$, where Y′ is the Young diagram obtained from Y by slicing off the first column. Thus Y′ has $k-1$ columns, so by the induction hypothesis the rest of the symmetry operators in P_{Y} can be connected in exactly one nonvanishing way (up to an overall sign).

Orthogonality

If Y_a and Y_b denote standard tableaux derived from the same Young diagram Y, then $P_{\mathrm{Y}_a}P_{\mathrm{Y}_b}=P_{\mathrm{Y}_b}P_{\mathrm{Y}_a}=\delta_{ab}P^2_{\mathrm{Y}_a}$, since there is a permutation of the lines connecting the symmetry operators of Y with those of Z and by uniqueness of the nonzero connection the result is either $P^2_{\mathrm{Y}_a}$ (if $\mathrm{Y}_a=\mathrm{Y}_b$) or (if $\mathrm{Y}_a\neq\mathrm{Y}_b$).

Next, consider two different Young diagrams Y and Z with the same number of boxes. Since at least one column must be bigger in (say) Y than in Z and the p lines from the corresponding antisymmetrizer must connect to different symmetrizers, it is not possible to make a nonzero connection between the antisymmetrizers of P_{Y_a} to the symmetrizers in P_z, where subscripts a and b denote any standard tableaux of Y and Z. Hence $P_{\mathrm{Y}_a}P_{\mathrm{Z}_b}=0$, and by a similar argument $P_{\mathrm{Z}_b}P_{\mathrm{Y}_a}=0$.

Normalization and completeness

We now derive the formula for the normalization factor α_{Y} such that the Young projection operators are idempotent, $P^2_{\mathrm{Y}_a}=P_{\mathrm{Y}_a}$. By the normalization of the symmetry operators, Young projection operators corresponding to fully symmetric or antisymmetric Young tableaux will be idempotent with $\alpha_{\mathrm{Y}}=1$.

Diagrammatically $P^2_{\mathrm{Y}a}$ is simply P_{Y_a} connected to P_{Y_a}, hence it may be viewed as a set of *outer* symmetry operators connected by a set of *inner* symmetry operators. Expanding all the inner symmetry operators and using the uniqueness of the nonzero connection between the symmetrizers and antisymmetrizers of the Young projection operators, we find that each term in the expansion is

either 0 or a copy of P_{Y_a}. For a Young diagram with s rows and t columns there will be a factor of $1/|S_i|!(1/|A_j|!)$ from the expansion of each inner (anti)symmetrizer, so we find

$$P_{Y_a}^2 = \alpha_{Y_a}^2$$
$$= \frac{\alpha_{Y_a}^2}{\prod_{i=1}^{s}|S_i|!\prod_{j=1}^{t}|A_j|!}\sum_{\sigma}\sigma$$
$$= \alpha_{Y_a}\frac{\kappa_Y}{\prod_{i=1}^{s}|S_i|!\prod_{j=1}^{t}|A_j|!}P_{Y_a},$$

where the sum is over permutation σ from the expansion of the inner symmetry operators. Note that by the uniqueness of the connection between the symmetrizers and antisymmetrizers, the constant κ_Y is independent of which tableau gives rise to the projection, and consequently the normalization constant α_Y depends only on the Young diagram and not the tableau.

For a given k, consider the Young projection operators P_{Y_a} corresponding to all the k-box Young tableaux. Since the operators P_{Y_a} are orthogonal and in one-to-one correspondence with the Young tableaux, it follows from the discussion in Sec. IV A that there are no other operators of k lines orthogonal to this set. Hence the P_{Y_a}'s form a complete set, so that

$$\mathbb{I} = \sum_{Y_a} P_{Y_a}. \tag{A1}$$

Expanding the projections the identity appears only once, so we have

$$P_{Y_a} = \alpha_Y \frac{1}{\prod_{i=1}^{s}|S_i|!\prod_{j=1}^{t}|A_j|!}\left(\underline{\equiv}_{p} + \ldots\right),$$

and using this, equation (A1) states

$$\underline{\equiv}_{p} = \left(k!\sum_{Y}\frac{\alpha_Y/|Y|}{\prod_{i=1}^{s}|S_i|!\prod_{j=1}^{t}|A_j|!}\right)\underline{\equiv}_{p} \tag{A2}$$

since all permutation different from the identity must cancel. When changing the sum from a sum over the tableaux to a sum over the Young diagrams we use that α_Y depends only on the diagram and that there are $\Delta_Y = k!/|Y|$ k-standard tableaux for a given diagram. Choosing

$$\alpha_Y = \frac{\Pi_{i=1}^{s}|S_i|!\Pi_{j=1}^{t}|A_j|!}{|Y|}, \tag{A3}$$

the factor on the right-hand side of (A2) is 1 by (8).

Since the choice of normalization (A3) gives the completeness relation (A1), it follows that it also gives idempotent operators, multiplying by P_{Z_b} on both sides of (A1) and using orthogonality, we find $P_{Z_b} = P_{Z_b}^2$ for any Young tableau Z_b.

Dimensionality

To prove the dimension formula (11) we need the identities

$$\underline{\equiv}_{p} = \frac{1}{p}\left(\underline{\equiv}_{p-1} + (p-1)\underline{\equiv}_{p-1}\right) \tag{A4}$$

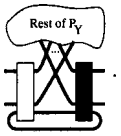
and

$$\underline{\equiv}_{p} = \frac{1}{p}\left(\underline{\equiv}_{p-1} - (p-1)\underline{\equiv}_{p-1}\right) \tag{A5}$$

given in Ref. 18. For Young tableaux with a single row or column, the dimension formula can be derived directly using the relations (A4) and (A5).

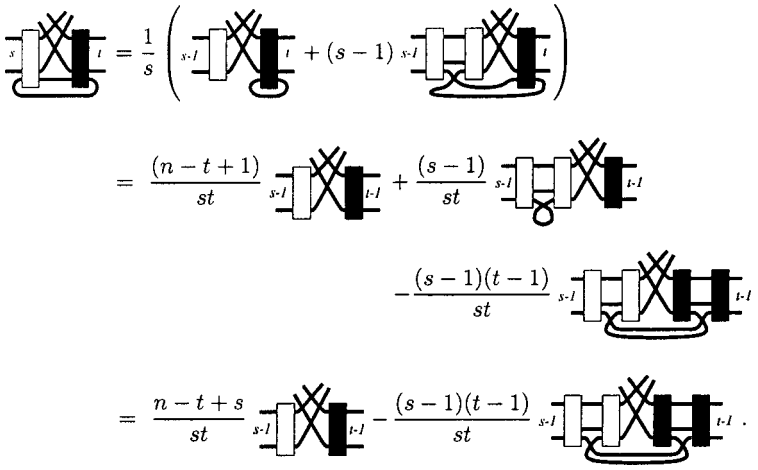
Let Y be a standard tableau with k boxes, and Y′ the standard Young tableau obtained from it by removing the box containing k. Draw the Young projection operators corresponding to Y and Y′ and note that P_{Y} with the "last" line traced is proportional to $P_{\mathrm{Y}'}$.

Quite generally the contraction of the last line will look like



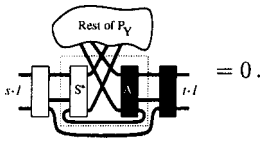
(A6)

Using (A4) and (A5) we have



Inserting this into (A6) we see that the first term is proportional to the projection operator $P_{\mathrm{Y}'}$

The second term vanishes,



If we ignore the internal structure within the dotted box we see that this is exactly of the form of $P_{\mathrm{Y}'}$, except that the "last" symmetrizers and antisymmetrizers are connected by a line. There is a unique nonvanishing way of connecting the symmetrizers and antisymmetrizers in $P_{\mathrm{Y}'}$, and the "last" symmetrizer and antisymmetrizer are not connected in this, as they correspond to a row and column with no common box in the Young tableau. Therefore every term obtained from the expansion of the dotted box must vanish.

The dimensionality formula follows by induction on the number of boxes in the Young diagrams with the dimension of a single box Young diagram being n. Let Y be a Young diagram with p boxes. We assume that the dimensionality formula is valid for any Young diagram with $p-1$ boxes. With $P_{\mathrm{Y}'}$ obtained from P_{Y} as above, we have (using the above calculation and writing D_{Y} for the diagrammatic part of P_{Y})

$$\dim P_{\mathrm{Y}} = \alpha_{\mathrm{Y}}\ \mathrm{tr}\ D_{\mathrm{Y}} = \frac{n-t+s}{st}\alpha_{\mathrm{Y}}\ \mathrm{tr}\ D_{\mathrm{Y}'} = (n-t+s)\alpha_{Y'}\frac{|\mathrm{Y}'|}{|\mathrm{Y}|}\mathrm{tr}\ D_{\mathrm{Y}'} = (n-t+s)\frac{f_{\mathrm{Y}'}}{|\mathrm{Y}|} = \frac{f_{\mathrm{Y}}}{|\mathrm{Y}|}.$$

This completes the proof of the dimensionality formula (11).

[1] C. G. J. Jacobi, "De functionibus alternantibus earumque divisione per productum e differentiis elementorum conflatum," *Collected Works* [J. Reine Angew. Math. **22**, 43 (1841)].
[2] I. Schur, "Uber eine klasse von matrizen, die sich einer gegeben matrix zuordnen lasses", in *Gesammelte Abhandlungen*,

(Springer, Berlin, 1973), pp. 1–71.
[3] A. Young, *The Collected Papers of Alfred Young* (University of Toronto Press, Toronto, 1977), with the eight articles on quantitative substitutional analysis, including Proc. London Math. Soc. **33**, 97 (1900); **28**, 255 (1928); **31**, 253 (1930).
[4] E. Weisstein, Young tableaux, http://mathworld.wolfram.com/YoungTableau.html.
[5] H. Weyl, *The Theory of Groups and Quantum Mechanics* (Methuen, London, 1931), translated by H. P. Robertson.
[6] M. Hamermesh, *Group Theory and Its Application to Physical Problems* (Dover, New York, 1962).
[7] A. Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1962), Vol. 2.
[8] A. J. Coleman, Adv. Quantum Chem. **4**, 83 (1968).
[9] D. B. Lichtenberg, *Unitary Symmetry and Elementary Particles* (Academic, New York, 1970).
[10] B. G. Wybourne, *Symmetry Principles and Atomic Spectroscopy* (Wiley, New York, 1970).
[11] H. Georgi, *Lie Algebras in Particle Physics* (Perseus, Reading, MA,1999).
[12] H. Boerner, *Representations of Groups* (North-Holland, Amsterdam, 1970).
[13] G. E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and its Applications (Addison-Wesley, NewYork, 1976).
[14] R. P. Stanley, *Enumerative Combinations* (MIT, Cambridge, 1999); http://www-math.mit.edu/~rstan/ec/
[15] W. Fulton and J. Harris, *Representation Theory*, Graduate Texts in Mathematics (Springer, Berlin, 1991).
[16] W. Fulton, *Young Table, With Applications to Representation Theory and Geometry* (MIT, Cambridge, 1999).
[17] L. Manivel, *Symmetric Functions, Schubert Polynomials and Degeneracy Loci* (American Mathematical Society, Providence, RI, 2001); http://www-fourier.ujf-grenoble.fr/~manivel/cours.html
[18] P. Cvitanović, *Group Theory* (Princeton University Press, Princeton, NJ, 2004); http://www.nbi.dk/group theory
[19] R. Penrose, *Applications of Negative Dimensional Tensors* (Academic, New York, 1971), pp.221–244.
[20] R. Penrose and M. A. H. MacCallum, Phys. Lett., C **65**, 1 (1973).
[21] R. Penrose, *Angular Momentum: An Approach to Combinatorial Space-time* (MIT, Cambridge, 1971).
[22] G. P. Canning, Phys. Rev. D **18**, 395 (1978).
[23] J. B. Mandula, Diagrammatic techniques in group theory, 1981, notes taken by S. N. Coulson and A. J. G Hey, University of Southampton.
[24] G. E. Stedman, *Diagram Techniques in Group Theory* (MIT, Cambridge, 1990).
[25] A. D. Kennedy, *Group Algebras, Lie Algebras, and Clifford Algebras*, Colloquium (Moscow State University, Moscow, 1997) http://www.ph.ed.ac.uk/~adk/algebra-slides
[26] H. Elvang, Birdtracks, Young projections, colours, 1999; www.nbi.dk/~elvang/rerep.ps, MPhys project in Mathematical Physics.
[27] J. S. Frame, D. de B. Robinson, and R. M. Thrall, Can. J. Math. **6**, 316 (1954).
[28] D. de B. Robinson, *Representation Theory of the Symmetric Group* (University of Toronto Press, Toronto, 1961).
[29] G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Encyclopedia of Mathematics and Its Applications (Addison-Wesley, Reading, MA, 1981).
[30] R. Goodman and N. R. Wallach, *Representations and Invariants of the Classical Groups* (MIT, Cambridge, 1998).
[31] C. Bessenrodt, Ann. Comb. **2**, 103 (1998), The hook theorem follows directly from Theorem 1.1.
[32] B. E. Sagan, *The Symmetric Group* (Springer-Verlag, New York, 2001).
[33] B. L. van der Waerden, *Algebra* 4th ed. (Springer-Verlag, Berlin, 1959), Vol. 2.
[34] G. t' Hooft and M. Veltman, DIAGRAMMAR, CERN Report 73/9 (1973); http://www.library.uu.n1/digiarchief/dip/dispute/2002-0718-131954/Diagrammar - 1973.pdf
[35] E. P. Wigner, *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* (Academic, New York, 1959).
[36] P. Cvitanović and A. D. Kennedy, Phys. Scr. **26**, 5 (1982).