Group Theory

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Lie's, Tracks, and Exceptional Groups

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dedicated to the memory of
Boris Weisfeiler and William E. Caswell

Contents

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The appellation "birdtracks" is due to Bernice Durand who entered my office, saw the blackboard and asked puzzled: "What is this? Footprints left by birds scurrying along a sandy beach?"

I am grateful to Dorte Glass for typing most of the manuscript, to the Aksel Tovborg Jensens Legat, Kongelige Danske Videnskabernes Selskab, for financial support which made the transformation from hand-drawn to drafted birdtracks possible, and, most of all, to the good Anders Johansen who did most of drawing of the birdtracks and without whom this book would still be but a collection of squiggles in my filing cabinet. Carol Monsrud and Cecile Gourgues helped with typing the early version of this manuscript.

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I would love to thank W.E. Caswell for a perspicacious observation, and B. Weisfeiler for delightful discussions at I.A.S, but it cannot be done. In 1985, while hiking in Andes, Boris was allegedly kidnapped by the Chilean state security, and tortured and executed by Nazis at Colonia Dignidad, Chile. Bill boarded flight AA77 on September 11, 2001 and was flown into Pentagon by a no less charming group of Islamic fanatics. This book is dedicated to their memory.

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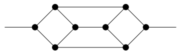
Chapter One

Introduction

This monograph offers a derivation of all classical and exceptional semi-simple Lie algebras through a classification of "primitive invariants". Using somewhat unconventional notation inspired by the Feynman diagrams of quantum field theory, the invariant tensors are represented by diagrams; severe limits on what simple groups could possibly exist are deduced by requiring that irreducible representations be of integer dimension. The method provides the full Killing-Cartan list of all possible simple Lie algebras, but fails to prove the existence of F_4, E_6, E_7 and E_8.

One simple quantum field theory question started this project; what is the group theoretic factor for the following Quantum Chromodynamics gluon self-energy diagram

 $= ?$ (1.1)

I first computed the answer for $SU(n)$. There was a hard way of doing it, using Gell-Mann f_{ijk} and d_{ijk} coefficients. There was also an easy way, where one could doodle oneself to the answer in a few lines. This is the "birdtracks" method which will be described here. It works nicely for $SO(n)$ and $Sp(n)$ as well. Out of curiosity, I wanted the answer for the remaining five exceptional groups. This engendered further thought, and that which I learned can be better understood as the answer to a different question. Suppose someone came into your office and asked, "On planet Z, mesons consist of quarks and antiquarks, but baryons contain three quarks in a symmetric color combination. What is the color group?" The answer is neither trivial, nor without some beauty (planet Z quarks can come in 27 colors, and the color group can be E_6).

Once you know how to answer such group-theoretical questions, you can answer many others. This monograph tells you how. Like the brain, it is divided into two halves; the plodding half and the interesting half.

The plodding half describes how group theoretic calculations are carried out for unitary, orthogonal and symplectic groups, chapters 3–15. Except for the "negative dimensions" of chapter 13 and the "spinsters" of chapter 14, none of that is new, but the methods are helpful in carrying out daily chores, such as evaluating Quantum Chromodynamics group theoretic weights, evaluating lattice gauge theory group integrals, computing $1/N$ corrections, evaluating spinor traces, evaluating casimirs, implementing evaluation algorithms on computers, and so on.

The interesting half, chapters 16–21, describes the "exceptional magic" (a new construction of exceptional Lie algebras), the "negative dimensions" (relations between bosonic and fermionic dimensions). Open problems and personal confessions are relegated to the epilogue, sect. 21.3. The methods used are applicable to field

theoretic model building. Regardless of their potential applications, the results are sufficiently intriguing to justity this entire undertaking. In what follows we shall forget about quarks and quantum field theory, and offer instead a somewhat unorthodox introduction to the theory of Lie algebras. If the style is not Bourbaki, it is not so by accident.

There are two complementary approaches to group theory. In the *canonical* approach one chooses the basis, or the Clebsch-Gordan coefficients, as simply as possible. This is the method which Killing [97] and Cartan [20] used to obtain the complete classification of semi-simple Lie algebras, and which has been brought to perfection by Dynkin [57]. There exist many excellent reviews of applications of Dynkin diagram methods to physics, such as the review by Slansky [145].

In the *tensorial* approach pursued here, the bases are arbitrary and every statement is invariant under change of basis. Tensor calculus deals directly with the invariant blocks of the theory and gives the explicit forms of the invariants, Clebsch-Gordan series, evaluation algorithms for group theoretic weights, *etc*.

The canonical approach is often impractical for computational purposes, as a choice of basis requires a specific coordinatization of the representation space. Usually, nothing that we want to compute depends on such a coordinatization; physical predictions are pure scalar numbers ("color singlets"), with all tensorial indices summed over. However, the canonical approach can be very useful in determining chains of subgroup embeddings, we refer the reader to the Slansky review [145] for such applications. Here we shall concentrate on tensorial methods, borrowing from Cartan and Dynkin only the nomenclature for identifying irreducible representations. Extensive listings of these are given by McKay and Patera [109] and Slansky [145].

To appreciate the sense in which canonical methods are impractical, let us consider using them to evaluate the group-theoretic factor associated with diagram (1.1) for the exceptional group E_8. This would involve summations over 8 structure constants. The Cartan-Dynkin construction enables us to construct them explicitly; an E_8 structure constant has about $248^3/6$ elements, and the direct evaluation of the group-theoretic factor for diagram (1.1) is tedious even on a computer. An evaluation in terms of a canonical basis would be equally tedious for $SU(16)$; however, the tensorial approach illustrated by the example of sect. 2.2 yields the answer for all $SU(n)$ in a few steps.

Simplicity of such calculations is one motivation for formulating a tensorial approach to exceptional groups. The other is the desire to understand their geometrical significance. The Killing-Cartan classification is based on a mapping of Lie algebras onto a Diophantine problem on the Cartan root lattice. This yields an exhaustive classification of simple Lie algebras, but gives no insight into the associated geometries. In the 19th century, the geometries or the invariant theory were the central question and Cartan, in his 1894 thesis, made an attempt to identify the primitive invariants. Most of the entries in his classification were the classical groups $SU(n)$, $SO(n)$ and $Sp(n)$. Of the five exceptional algebras, Cartan [21] identified G_2 as the group of octonion isomorphisms, and noted already in his thesis that E_7 has a skew-symmetric quadratic and a symmetric quartic invariant. Dickinson [51] characterized E_6 as a 27-dimensional group with a cubic invariant. The fact that the orthogonal, unitary and symplectic groups were invariance groups of real, complex and quaternion norms suggested that the exceptional groups were associated with octonions, but it took more than fifty years to establish this connection. The remaining four exceptional Lie algebras emerged as rather complicated constructions from octonions and Jordan algebras, known as the Freudenthal-Tits construction. A mathematician's history of this subject is given in a delightful review by Freudenthal [72]. The problem has been taken up by physicists twice, first by Jordan, von Neumann and Wigner [88], and then in the 1970's by Gürsey and collaborators [78, 79]. Jordan *et al.'s* effort was a failed attempt at formulating a new quantum mechanics which would explain the neutron, discovered in 1932. However, it gave rise to the Jordan algebras, which became a mathematics field in itself. Gürsey *et al.* took up the subject again in the hope of formulating a quantum mechanics of quark confinement; however, the main applications so far have been in building models of grand unification.

Although beautiful, the Freudenthal-Tits construction is still not practical for the evaluation of group-theoretic weights. The reason is this: the construction involves $[3{\times}3]$ octonian matrices with octonian coefficients, and the 248 dimensional defining space of E_8 is written as a direct sum of various subspaces. This is convenient for studying subgroup embeddings [138], but awkward for group-theoretical computations.

The inspiration for the primitive invariants construction came from the axiomatic approach of Springer [146, 147] and Brown [15]: one treats the defining representation as a single vector space, and characterizes the primitive invariants by algebraic identities. This approach solves the problem of formulating efficient tensorial algorithms for evaluating group-theoretic weights and it yields some intuition about the geometrical significance of the exceptional Lie groups. Such intuition might be of use to quark-model builders. For example, because $SU(3)$ has a cubic invariant $\epsilon^{abc}q_a q_b q_c$, Quantum Chromodynamics, based on this color group, can accommodate 3-quark baryons. Are there any other groups that could accommodate 3-quark singlets? As we shall show, G_2, F_4 and E_6 are some of the groups whose defining representations possess such invariants.

Beyond its utility as a computational technique, the primitive invariants construction of exceptional groups yields several unexpected results. First, it generates in a somewhat magical fashion a triangular array of Lie algebras, depicted in fig. 1.1. This is a classification of Lie algebras different from Cartan's classification; in this new classification, all exceptional Lie groups appear in the same series (the bottom line of fig. 1.1). The second unexpected result is that many groups and group representations are mutually related by interchanges of symmetrizations and antisymmetrizations and replacement of the dimension parameter n by $-n$. I call this phenomenon "negative dimensions".

For me, the greatest surprise of all is that in spite of all the magic and the strange diagrammatic notation, the resulting manuscript is in essence not very different from Wigner's [158] classic group theory book. Regardless of whether one is doing atomic, nuclear or particle physics, all physical predictions ("spectroscopic levels") are expressed in terms of Wigner's $3n$-j coefficients, which can be evaluated by means of recursive or combinatorial algorithms.

						0 / 0	3 / $\mathbf{A}_1$ / 2
					0 / 0	1 / $\mathbf{U}(1)$ / 1	8 / $\mathbf{A}_2$ / 3
				0 / 0	0 / 1	3 / $\mathbf{A}_1$ / 3	14 / $\mathbf{G}_2$ / 7
			0 / 0	0 / 1	2 / $2\mathbf{U}(1)$ / 2	9 / $\mathbf{3A}_1$ / 4	28 / $\mathbf{D}_4$ / 8
		0 / 0	0 / 2	3 / $\mathbf{A}_1$ / 5	8 / $\mathbf{A}_2$ / 8	21 / $\mathbf{C}_3$ / 14	52 / $\mathbf{F}_4$ / 26
	0 / 0	0 / 1	2 / $2\mathbf{U}(1)$ / 3	8 / $\mathbf{A}_2$ / 6	16 / $\mathbf{2A}_2$ / 9	35 / $\mathbf{A}_5$ / 15	78 / $\mathbf{E}_6$ / 27
0 / 0	1 / $\mathbf{U}(1)$ / 2	3 / $\mathbf{A}_1$ / 4	9 / $\mathbf{3A}_1$ / 8	21 / $\mathbf{C}_3$ / 14	35 / $\mathbf{A}_5$ / 20	66 / $\mathbf{D}_6$ / 32	133 / $\mathbf{E}_7$ / 56
3 / $\mathbf{A}_1$ / 3	8 / $\mathbf{A}_2$ / 8	14 / $\mathbf{G}_2$ / 14	28 / $\mathbf{D}_4$ / 28	52 / $\mathbf{F}_4$ / 52	78 / $\mathbf{E}_6$ / 78	133 / $\mathbf{E}_7$ / 133	248 / $\mathbf{E}_8$ / 248

Figure 1.1 The "magic triangle" for Lie algebras. The Freudenthal "magic square" is marked by the dotted line. The number in the lower left corner of each entry is the dimension of the defining representation. For more details consult chapter 21.

Chapter Two

A preview

The theory of Lie groups presented here had mutated greatly throughout its genesis. It arose from concrete calculations motivated by physical problems; but as it was written, the generalities were collected into introductory chapters, and the applications receded later and later into the text.

As a result, the first seven chapters are largely a compilation of definitions and general results which might appear unmotivated on first reading. The reader is advised to work through the examples, sect. 2.2 and sect. 2.3 in this chapter, jump to the topic of possible interest (such as the unitary groups, chapter 9, or the E_8 family, chapter 17), and birdtrack if able or backtrack when necessary.

The goal of these notes is to provide the reader with a set of basic group-theoretic tools. They are not particularly sophisticated, and they rest on a few simple ideas. The text is long, because various notational conventions, examples, special cases and applications have been laid out in detail, but the basic concepts can be stated in a few lines. We shall briefly state them in this chapter, together with several illustrative examples. This preview presumes that the reader has considerable prior exposure to group theory; if a concept is unfamiliar, the reader is referred to the appropriate section for a detailed discussion.

2.1 BASIC CONCEPTS

An average quantum theory is constructed from a few building blocks, which we shall refer to as the *defining rep*. They form the defining multiplet of the theory - for example, the "quark wave functions" q_a. The group-theoretical problem consists of determining the symmetry group, *ie.* the group of all linear transformations

$$q'_a = G_a{}^b q_b \qquad a, b = 1, 2, \ldots, n \,,$$

which leaves invariant the predictions of the theory. The $[n \times n]$ matrices G form the *defining rep* of the invariance group $\mathcal{G}$. The conjugate multiplet ("antiquarks") transforms as

$$q'^a = G^a{}_b q^b \,.$$

Combinations of quarks and antiquarks transform as *tensors*, such as

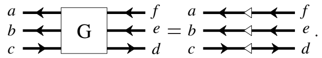
$$p'_a q'_b r'^c = G_{a\,b,\,d}^{\;\;c\;\;\,ef} p_f q_e r^d \,,$$
$$G_{a\,b,\,d}^{\;\;c\;\;\,ef} = G_a{}^f G_b{}^e G^c{}_d \,.$$

(see sect. 3.1.4). Tensor reps are plagued by a proliferation of indices. These indices can either be replaced by a few collective indices

$$\alpha = \left\{ \begin{matrix} c \\ ab \end{matrix} \right\} \,, \qquad \beta = \left\{ \begin{matrix} ef \\ d \end{matrix} \right\} \,,$$

$$q'_\alpha = G_\alpha{}^\beta q_\beta \,, \tag{2.1}$$

or represented diagrammatically



(Diagrammatic notation is explained in sect. 4.1). Collective indices are convenient for stating general theorems; diagrammatic notation speeds up explicit calculations.

A polynomial

$$H(\overline{q}, \overline{r}, s, \ldots) = h_{ab\ldots}{}^{\ldots c}\, q^a r^b \ldots s_c$$

is an invariant if (and only if) for any transformation $G \in \mathcal{G}$ and for any set of vectors $q, r, s, \ldots$ (see sect. 3.3)

$$H(\overline{Gq}, \overline{Gr}, Gs, \ldots) = H(\overline{q}, \overline{r}, s, \ldots) \,. \tag{2.2}$$

An invariance group is defined by its *primitive invariants*, *ie.* by a list of the elementary "singlets" of the theory. For example, the orthogonal group $O(n)$ is defined as the group of all transformations which leaves the length of a vector invariant (see chapter 10). Another example is the color $SU(3)$ of QCD which leaves invariant the mesons $(q\bar{q})$ and the baryons (qqq) (see sect. 15.2). A complete list of primitive invariants *defines* the invariance group via the invariance conditions (2.2); only those transformations, which respect them, are allowed.

It is not necessary to list explicitly the components of primitive invariant tensors in order to define them. For example, the $O(n)$ group is defined by the requirement that it leaves invariant a symmetric and invertible tensor $g_{ab} = g_{ba}$, $\det(g) \neq 0$. Such definition is basis independent, while a component definition $g_{11} = 1, g_{12} = 0, g_{22} = 1, \ldots$ relies on a specific basis choice. We shall define all simple Lie groups in this manner, specifying the primitive invariants only by their symmetry, and by the basis-independent algebraic relations that they must satisfy.

These algebraic relations (which we shall call *primitiveness conditions*) are hard to describe without first giving some examples. In their essence they are statements of irreducibility; for example, if the primitive invariant tensors are δ^a_b, h_{abc} and h^{abc}, then $h_{abc}h^{cbe}$ must be proportional to δ^e_a, as otherwise the defining rep would be reducible. (Reducibility is discussed in sect. 3.4, sect. 3.5 and chapter 5).

The objective of physicist's group-theoretic calculations is a description of the spectroscopy of a given theory. This entails identifying the levels (irreducible multiplets), the degeneracy of a given level (dimension of the multiplet) and the level splittings (eigenvalues of various casimirs). The basic idea, that enables us to carry this program through, is extremely simple: a hermitian matrix can be diagonalized. This fact has many names: Schur's lemma, Wigner-Eckart theorem, full reducibility of unitary reps, and so on (see sect. 3.4 and sect. 5.3). We exploit it by constructing invariant hermitian matrices M from the primitive invariant tensors. M's have collective indices (2.1) and act on tensors. Being hermitian, they can be diagonalized

$$CMC^\dagger = \begin{pmatrix} \lambda_1 & 0 & 0 & & \dots \\ 0 & \lambda_1 & 0 & & \\ 0 & 0 & \lambda_1 & & \\ & & & \lambda_2 & \\ \vdots & & & & \ddots \end{pmatrix},$$

and their eigenvalues can be used to construct projection operators which reduce multiparticle states into direct sums of lower-dimensional reps (see sect. 3.4):

$$P_i = \prod_{j \neq i} \frac{M - \lambda_j \mathbf{1}}{\lambda_i - \lambda_j} = C^\dagger \begin{pmatrix} \begin{array}{cc|} \ddots & \vdots \\ \dots & 0 \\ \hline \end{array} & \dots & 0 \\ \vdots & \boxed{\begin{matrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & 1 \end{matrix}} & \vdots \\ 0 & \dots & \begin{array}{|cc} \hline 0 & \dots \\ \vdots & \ddots \end{array} \end{pmatrix} C \,. \tag{2.3}$$

An explicit expression for the diagonalizing matrix C (Clebsch-Gordan coefficients, sect. 4.2) is unnecessary – it is in fact often more of an impediment than an aid, as it obscures the combinatorial nature of group theoretic computations (see sect. 4.7).

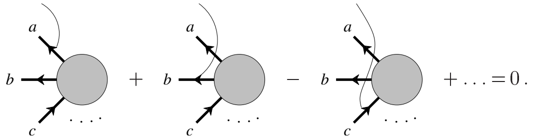
All that is needed in practice is knowledge of the characteristic equation for the invariant matrix M (see sect. 3.4). The characteristic equation is usually a simple consequence of the algebraic relations satisfied by the primitive invariants, and the eigenvalues λ_i are easily determined. λ_i' s determine the projection operators P_i, which in turn contain all relevant spectroscopic information: the rep dimension is given by $\mathrm{tr}\, P_i$, and the casimirs, 6-j's, crossing matrices and recoupling coefficients (see chapter 5) are traces of various combinations of P_i's. All these numbers are *combinatoric*; they can often be interpreted as the number of different colorings of a graph, the number of singlets, and so on.

The invariance group is determined by considering infinitesimal transformations

$$G_a^b \simeq \delta_b^a + i\epsilon_i (T_i)_a^b \,.$$

The generators T_i are themselves clebsches, elements of the diagonalizing matrix C for the tensor product of the defining rep and its conjugate. They project out the adjoint rep and are constrained to satisfy the *invariance conditions* (2.2) for infinitesimal transformations (see sect. 4.4 and sect. 4.5):

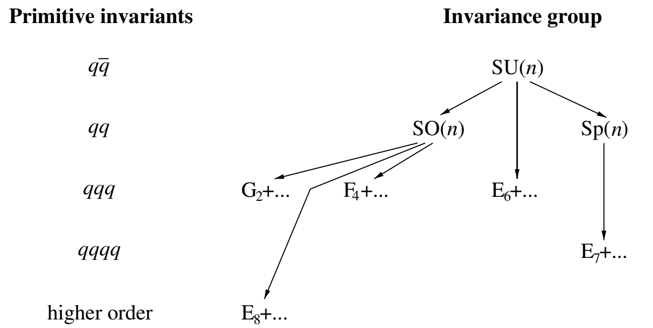
$$(T_i)_a^{a'} h_{a'b\dots}{}^{c\dots} + (T_i)_b^{b'} h_{ab'\dots}{}^{c\dots} - (T_i)_{c'}^{c} h_{ab\dots}{}^{c'\dots} + \dots = 0$$



(2.4)

As the corresponding projector operators are already known, we have an explicit construction of the symmetry group (at least infinitesimally – we will not consider discrete transformations).

If the primitive invariants are bilinear, the above procedure leads to the familiar tensor reps of classical groups. However, for trilinear or higher invariants the results are more surprising. In particular, all exceptional Lie groups emerge in a pattern of solutions which we will refer to as a "magic triangle". The logic of the construction can be schematically indicated by the following chains of subgroups (see chapter 16):



In the above diagram the arrows indicate the primitive invariants which characterize a particular group. For example, E_7 primitives are a sesquilinear invariant $q\bar{q}$, a skew symmetric qp invariant and a symmetric $qqqq$ (see chapter 20).

The strategy is to introduce the invariants one by one, and study the way in which they split up previously irreducible reps. The first invariant might be realizable in many dimensions. When the next invariant is added (sect. 3.5), the group of invariance transformations of the first invariant splits into two subsets; those transformations which preserve the new invariant, and those which do not. Such decompositions yield Diophantine conditions on rep dimensions. These conditions are so constraining that they limit the possibilities to a few which can be easily identified.

To summarize; in the primitive invariants approach, all simple Lie groups, classical as well as exceptional, are constructed by (see chapter 21):

i) defining a symmetry group by specifying a list of *primitive invariants*,

ii) using *primitiveness* and *invariance* conditions to obtain algebraic relations between primitive invariants,

iii) constructing *invariant matrices* acting on tensor product spaces,

iv) constructing *projection operators* for reduced rep from characteristic equations for invariant matrices.

Once the projection operators are known, all interesting spectroscopic numbers can be evaluated.

The foregoing run through the basic concepts was inevitably obscure. Perhaps working through the next two examples will make things clearer. The first example illustrates computations with classical groups. The second example is more interesting; it is a sketch of construction of irreducible reps of E_6.

2.2 FIRST EXAMPLE: $SU(N)$

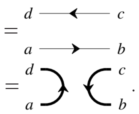
How do we describe the invariance group that preserves the norm of a complex vector? The *list of primitives* consists of a single primitive invariant

$$m(p,q) = \delta^a_b p^b q_a = \sum_{a=1}^{n} (p_a)^* q_a \,.$$

The Kronecker δ^a_b is the only primitive invariant tensor. We can immediately write down the two *invariant matrices* on the tensor product of the defining space and its conjugate:

$$\text{identity}: \mathbf{1}_{d,b}^{a\ c} = \delta^a_b \delta^c_d =$$

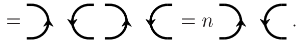
$$\text{trace}: T_{d,b}^{a\ c} = \delta^a_d \delta^c_b = \,.$$



The *characteristic equation* for T written out in the matrix, tensor and birdtrack notations is

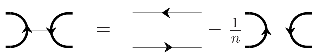
$$T^2 = nT$$

$$T_{d,e}^{a\ f} T_{f,b}^{e\ c} = \delta^a_d \delta^f_e \delta^e_f \delta^c_b = n\, T_{d,b}^{a\ c}$$



Here we have used $\delta^e_e = n$, the dimension of the defining vector space. The roots are $\lambda_1 = 0$, $\lambda_2 = n$, and the corresponding *projection operators* are

$$SU(n) \text{ adjoint rep:} \qquad P_1 = \frac{T-n\mathbf{1}}{0-n} = \mathbf{1} - \frac{1}{n}T$$



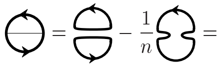
$$(2.5)$$

$$U(n) \text{ singlet:} \qquad P_2 = \frac{T-0\cdot\mathbf{1}}{n-1} = \frac{1}{n}T = \frac{1}{n}$$

Now we can evaluate any number associated with the $SU(n)$ adjoint rep, such as its dimension and various casimirs.

The *dimensions* of the two reps are computed by tracing the corresponding projection operators (see sect. 3.4)

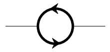
$$SU(n) \text{ adjoint: } d_1 = \text{tr}\, P_1 = \cdots = \delta^b_b \delta^a_a - \frac{1}{n}\delta^b_a \delta^a_b = n^2 - 1$$



$$\text{singlet: } d_2 = \text{tr}\, P_2 = \frac{1}{n}\cdots = 1\,.$$

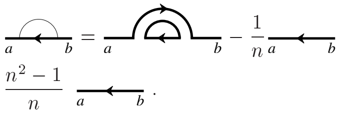
To evaluate *casimirs*, we need to fix the overall normalization of the generators of $SU(n)$. Our convention is to take

$$\delta_{ij} = \text{tr}\, T_i T_j = \,.$$



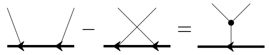
The value of the quadratic casimir for the defining rep is computed by substituting the adjoint projection operator

$$SU(n):\quad C_F \delta^b_a = (T_i T_i)^b_a = \cdots = \frac{n^2-1}{n} \,. \qquad (2.6)$$



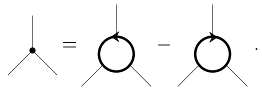
In order to evaluate the quadratic casimir for the adjoint rep, we need to replace the structure constants iC_{ijk} by their *Lie algebra* definition (see sect. 4.5)

$$T_i T_j - T_j T_i = iC_{ijk}$$



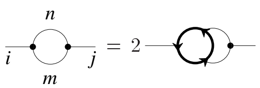
Tracing with T_k, we can express C_{ijk} in terms of the defining rep traces:

$$iC_{ijk} = \text{tr}\,(T_i T_j T_k) - \text{tr}\,(T_j T_i T_k)$$

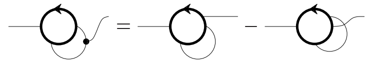


The adjoint quadratic casimir $C_{imn}C^{nmj}$ is now evaluated by first eliminating C_{ijk}'s in favor of the defining rep:

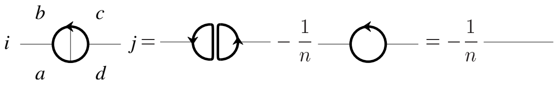
$$\delta_{ij} C_A = \cdots = 2 \cdots \,.$$



The remaining C_{ijk} can be unwound by the Lie algebra commutator



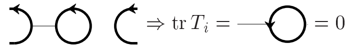
We have already evaluated the quadratic casimir (2.6) in the first term. The second term we evaluate by substituting the adjoint projection operator



$$\text{tr}\,(T_i T_k T_j T_k) = (T_i)^b_a (P_1)_{d,\ b}^{a,\ c} (T_j)^d_c = (T_i)^a_a (T_j)^c_c - \frac{1}{n}(T_i)^b_a (T_j)^a_b \,.$$

The $(T_i)^a_a (T_j)^c_c$ term vanishes by the tracelessness of T_i's. This can be considered a consequence of the orthonormality of the two projection operators P_1 and P_2 in (2.5) (see (3.47)):

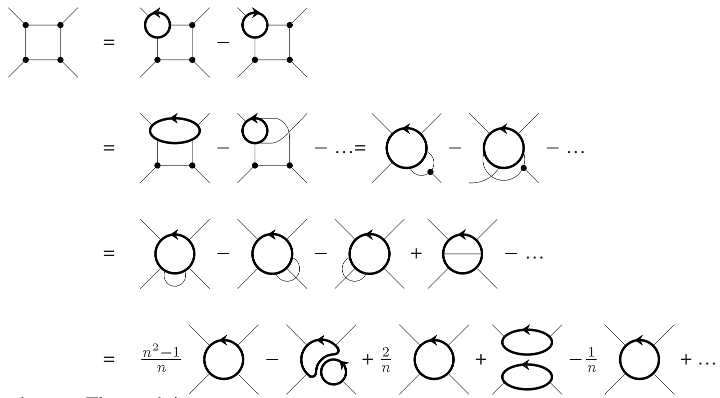
$$0 = P_1 P_2 = \cdots \Rightarrow \text{tr}\, T_i = \cdots = 0$$



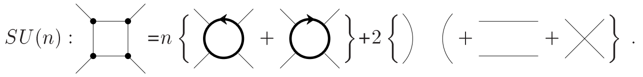
Combining the above expressions we finally obtain

$$C_A = 2\left(\frac{n^2-1}{n} + \frac{1}{n}\right) = 2n \ .$$

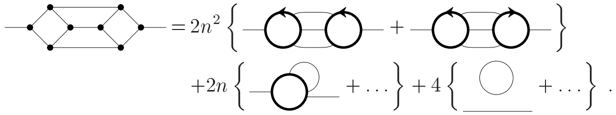
The problem (1.1) that started all this is evaluated the same way. First we relate the adjoint quartic casimir to the defining casimirs:



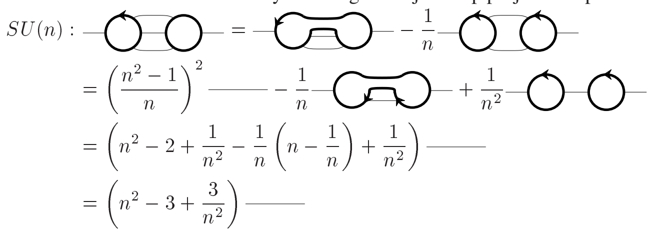
and so on. The result is



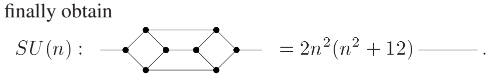
(1.1) is now reexpressed in terms of the defining rep casimirs:



The first two terms are evaluated by inserting the adjoint rep projection operators



and the remaining terms have already been evaluated. Collecting everything together, we finally obtain



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This example was unavoidably lengthy; the main point is that the evaluation is performed by a substitution algorithm and is easily automated. Any graph, no matter how complicated, is eventually reduced to a polynomial in traces of $\delta^a_a = n$, *ie.* the dimension of the defining rep.

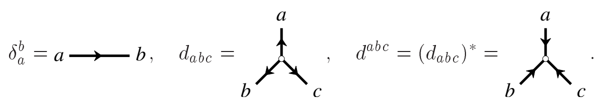
2.3 SECOND EXAMPLE: E_6 FAMILY

What invariance group preserves norms of complex vectors, as well as a symmetric cubic invariant

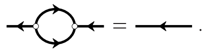
$$D(p,q,r) = d^{abc} p_a q_b r_c = D(q,p,r) = D(p,r,q) \ ?$$

We analyze this case following the steps of the summary of sect. 2.1:

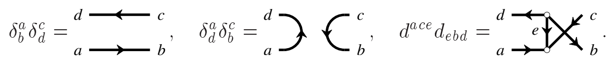
i) *primitive invariant tensors*:



ii) *primitiveness*: $d_{aef}d^{efb}$ must be proportional to δ^a_b, the only primitive 2-index tensor. We use this to fix the overall normalization of d_{abc}'s:



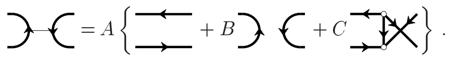
iii) *invariant hermitian matrices*: We shall construct here the adjoint rep projection operator on the tensor product space of the defining rep and its conjugate. All invariant matrices on this space are



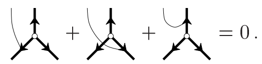
They are hermitian in the sense of being invariant under complex conjugation and transposition of indices (see (3.18)).

The adjoint projection operator must be expressible in terms of the four-index invariant tensors listed above:

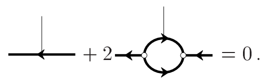
$$(T_i)^a_b (T_i)^d_c = A(\delta^a_c \delta^d_b + B\delta^a_b \delta^d_c + C d^{ade} d_{bce})$$



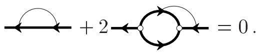
iv) *invariance*. The cubic invariant tensor satisfies (2.4)



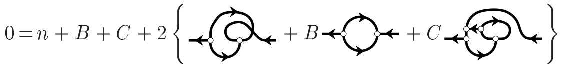
Contracting with d^{abc} we obtain



Contracting next with $(T_i)_a^b$, we get an invariance condition on the adjoint projection operator:



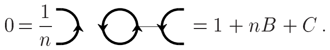
Substituting the adjoint projection operator yields the first relation between the coefficients in its expansion:



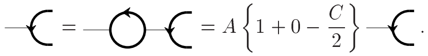
$$0 = B + C + \frac{n+2}{3} .$$

v) *the projection operators* should be orthonormal, $P_\mu P_\sigma = P_\mu \delta_{\mu\sigma}$. The adjoint projection operator is orthogonal to the singlet projection operator P, constructed in sect. 2.2. This yields the second relation on the coefficients:

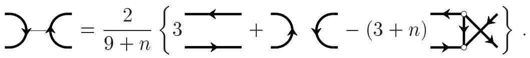
$$0 = P_A P_1$$



Finally, the overall normalization factor A is fixed by $P_A P_A = P_A$:

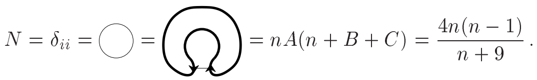


Combining the above 3 relations, we obtain the adjoint projection operator for the invariance group of a symmetric cubic invariant



The corresponding *characteristic equation*, mentioned in the point iv of the summary of sect. 2.1, is given in (18.10).

The dimension of the adjoint rep is obtained by tracing the projection operator



This *Diophantine condition* is satisfied by a small family of invariance groups, discussed in chapter 18. The most interesting member of this family is the exceptional Lie group E_6, with $n = 27$ and $N = 78$.

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Chapter Three

Invariants and reducibility

Basic group theoretic notions are introduced: groups, invariants, tensors, the diagrammatic notation for invariant tensors.

The basic idea is simple; a hermitian matrix can be diagonalized. If this matrix is an invariant matrix, it decomposes the reps of the group into direct sums of lower dimensional reps.

The key results are the construction of projection operators from invariant matrices (3.45), the Clebsch-Gordan coefficients rep of projection operators (4.16), the invariance conditions (4.33) and the Lie algebra relations (4.45).

3.1 PRELIMINARIES

In this section we define basic building blocks of the theory to be developed here: groups, vector spaces, algebras, *etc*. This material is covered in any introduction to group theory [152, 80]. Most of sect. 3.1.1 to sect. 3.1.4 is probably known to the reader and profitably skipped on the first reading.

3.1.1 Groups

Definition. A set of elements $g \in \mathcal{G}$ forms a group with respect to multiplication $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ if

(a) the set is *closed* with respect to multiplication; for any two elements $a, b \in \mathcal{G}$, the product $ab \in \mathcal{G}$.

(b) the multiplication is *associative*

$$(ab)c = a(bc)$$

for any three elements $a, b, c \in \mathcal{G}$.

(c) there exists an *identity* element $\mathbf{e} \in \mathcal{G}$ such that

$$\mathbf{e}g = g\mathbf{e} \qquad \text{for any } g \in \mathcal{G}\,.$$

(d) for any $g \in \mathcal{G}$ there exists an *inverse* g^{-1} such that

$$g^{-1}g = gg^{-1} = \mathbf{e}\,.$$

If the group is finite, the number of elements is called the *order* of the group and denoted $|\mathcal{G}|$.

If the multiplication $ab = ba$ is commutative for all $a, b \in \mathcal{G}$, the group is *abelian*.

Two groups with the same multiplication table are said to be *isomorphic*.

Definition. A *subgroup* $\mathcal{H} \leq \mathcal{G}$ is a subset of $\mathcal{G}$ that forms a group under multiplication. $\mathbf{e}$ is always a subgroup; so is $\mathcal{G}$ itself.

Definition. A *cyclic group* is a group generated from one of its elements, called the generator of the cyclic group. If n is the minimum integer such that $a^n = \mathbf{e}$, the set $\mathcal{G} = \{\mathbf{e}, a, a^2, \cdots, a^{n-1}\}$ is the cyclic group. As all elements commute, cyclic groups are abelian. Every subgroup of a cyclic group is cyclic.

3.1.2 Vector spaces

Definition. A set V of elements $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ is called a *vector* (or *linear*) *space* over a field $\mathbb{F}$ if

(a) *vector addition* "+" is defined in V such that V is an abelian group under addition, with identity element $\mathbf{0}$.

(b) the set is *closed* with respect to *scalar multiplication* and vector addition

$$\begin{aligned}
a(\mathbf{x}+\mathbf{y}) &= a\mathbf{x} + a\mathbf{y}\,, \qquad a, b \in \mathbb{F}, \quad \mathbf{x}, \mathbf{y} \in V \\
(a+b)\mathbf{x} &= a\mathbf{x} + b\mathbf{x} \\
a(b\mathbf{x}) &= (ab)\mathbf{x} \\
1\,\mathbf{x} &= \mathbf{x}\,, \qquad 0\,\mathbf{x} = \mathbf{0}\,.
\end{aligned}$$

Here the field $\mathbb{F}$ will be either $\mathbb{R}$, the field of reals numbers, or $\mathbb{C}$, the field of complex numbers (quaternion and octonion fields are discussed in sect. 16.6).

Definition. n-*dimensional complex vector space* V consists of all n-multiplets $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $x_i \in \mathbb{C}$. The two elements $\mathbf{x}, \mathbf{y}$ are equal if $x_i = y_i$ for all $0 \leq i \leq n$. The vector addition identity element is $\mathbf{0} = (0, 0, \cdots, 0)$.

Definition. A complex vector space V is an *inner product space* if, with every pair of elements $\mathbf{x}, \mathbf{y} \in V$, there is associated a unique *inner* (or *scalar*) product $(x, y) \in \mathbb{C}$, such that

$$\begin{aligned}
(x, y) &= (y, x)^* \\
(ax, by) &= a^* b(x, y)\,, \qquad a, b \in \mathbb{C} \\
(z, ax + by) &= a(z, x) + b(z, y)\,,
\end{aligned}$$

where * denotes complex conjugation.

Without any noteworthy loss of generality, we shall here define the scalar product of two elements of V by

$$(x, y) = \sum_{j=1}^{n} x_j^* y_j\,. \tag{3.1}$$

3.1.3 Algebra

Definition. A set of elements $\mathbf{t}_\alpha$ of a vector space $\mathcal{T}$ forms an algebra if, in addition to the vector addition and scalar multiplication;

(a) the set is *closed* with respect to multiplication $\mathcal{T} \cdot \mathcal{T} \to \mathcal{T}$, so that for any two elements $\mathbf{t}_\alpha, \mathbf{t}_\beta \in \mathcal{T}$, the product $\mathbf{t}_\alpha \cdot \mathbf{t}_\beta$ also belongs to $\mathcal{T}$:

$$\mathbf{t}_\alpha \cdot \mathbf{t}_\beta = \sum_{\gamma \in T} t_{\alpha\beta}{}^\gamma \mathbf{t}_\gamma \,. \tag{3.2}$$

(b) the multiplication operation is *bilinear*

$$\begin{aligned}(\mathbf{t}_\alpha + \mathbf{t}_\beta) \cdot \mathbf{t}_\gamma &= \mathbf{t}_\alpha \cdot \mathbf{t}_\gamma + \mathbf{t}_\beta \cdot \mathbf{t}_\gamma \\ \mathbf{t}_\alpha \cdot (\mathbf{t}_\beta + \mathbf{t}_\gamma) &= \mathbf{t}_\alpha \cdot \mathbf{t}_\beta + \mathbf{t}_\alpha \cdot \mathbf{t}_\gamma \,.\end{aligned}$$

The set of numbers $t_{\alpha\beta}{}^\gamma$ are called the *structure constants* of the algebra. They form a matrix rep of the algebra

$$(t_\alpha)_\beta{}^\gamma = t_{\alpha\beta}{}^\gamma \,, \tag{3.3}$$

whose dimension is the dimension of the algebra itself.

Depending on what further assumptions one makes on the multiplication, one obtains different types of algebras. For example, if the multiplication is associative

$$(\mathbf{t}_\alpha \cdot \mathbf{t}_\beta) \cdot \mathbf{t}_\gamma = \mathbf{t}_\alpha \cdot (\mathbf{t}_\beta \cdot \mathbf{t}_\gamma) \,,$$

the algebra is *associative*. Typical examples of products are the *matrix product*

$$(\mathbf{t}_\alpha \cdot \mathbf{t}_\beta)_a^c = (t_\alpha)_a^b (t_\beta)_b^c \qquad \mathbf{t}_\alpha \in V \otimes \bar{V} \,, \tag{3.4}$$

and the *Lie product*

$$(\mathbf{t}_\alpha \cdot \mathbf{t}_\beta)_a^c = (t_\alpha)_a^b (t_\beta)_b^c - (t_\alpha)_c^b (t_\beta)_b^a \qquad \mathbf{t}_\alpha \in V \otimes \bar{V} \,, \tag{3.5}$$

which defines a *Lie algebra*.

As a plethora of vector spaces, indices and conjugations looms large in our immediate future, it pays to streamline the notation now, by singling out one vector space as "defining", and replacing complex conjugation by raised indices.

3.1.4 Defining space, tensors, reps

Definition. Let V be the *defining* n-dimensional complex vector space. Associate with the defining n-dimensional complex vector space V a *conjugate* (or *dual*) n-dimensional vector space $\bar{V} = \{\bar{x} \mid \bar{x}^* \in V\}$ obtained by complex conjugation of elements $x \in V$. We shall denote the corresponding element of $\bar{V}$ by raising the index

$$x^a \;=\; (x_a)^* \,,$$

so the components of defining space vectors, resp. conjugate vectors, are distinguished by lower, resp. upper indices

$$\begin{aligned}x &= (x_1, x_2, \ldots, x_n) \,, \;\; x \in V \\ \bar{x} &= (x^1, x^2, \ldots, x^n) \,, \;\; \bar{x} \in \bar{V} \,.\end{aligned} \tag{3.6}$$

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Repeated index summation: Throughout this text, the repeated indices are always summed over

$$G_a^b x_b = \sum_{b=1}^{n} G_a^b x_b \,, \tag{3.7}$$

unless explicitly stated otherwise.

Definition. Let $\mathcal{G}$ be a group of transformations acting linearly on V, with the action of a group element $g \in \mathcal{G}$ on a vector $x \in V$ given by a unitary $[n \times n]$ matrix G

$$x'_a = G_a^b x_b \qquad a, b = 1, 2, \ldots, n \,. \tag{3.8}$$

We shall refer to G_a^b as the *defining rep* of the group. The action of $g \in \mathcal{G}$ on a vector $\bar{q} \in \bar{V}$ is given by the *conjugate rep* $G^\dagger$

$$x'^a = x^b (G^\dagger)_b^a \,, \qquad (G^\dagger)_b^a \equiv (G_a^b)^* \,. \tag{3.9}$$

Another way to distinguish G from $G^\dagger$ is to meticulously keep track of the relative ordering of the indices,

$$G_a^b \to G_a{}^b \,, \qquad (G^\dagger)_a^b \to G^b{}_a \,.$$

As we use $G^\dagger$ but occasionally, and keeping track of these indices is confusing enough as is, we desist. By defining the conjugate space $\bar{V}$ by complex conjugation and inner product (3.1), we have already chosen (without any loss of generality) δ_a^b as the invariant tensor with the bilinear form $(x, x) = x_b x^b$. From this choice it follows that, in the applications considered here, the group $\mathcal{G}$ is always assumed *unitary*

$$(G^\dagger)_a^c G_c^b = \delta_a^b \,. \tag{3.10}$$

Definition. A *tensor* $x \in V^p \otimes \bar{V}^q$ is any object that transforms under the action of $g \in \mathcal{G}$ as

$$x'^{a_1 a_2 \ldots a_q}_{b_1 \ldots b_p} = G^{a_1 a_2 \ldots a_q}_{b_1 \ldots b_p}{}_{,}{}^{d_p \ldots d_1}_{c_q \ldots c_2 c_1} \, x^{c_1 c_2 \ldots c_q}_{d_1 \ldots d_p} \,, \tag{3.11}$$

where the $V^p \otimes \bar{V}^q$ *tensor rep* of $g \in \mathcal{G}$ is defined by

$$G^{a_1 a_2 \ldots a_p}_{b_1 \ldots b_q}{}_{,}{}^{d_q \ldots d_1}_{c_p \ldots c_2 c_1} \equiv (G^\dagger)^{a_1}_{c_1} (G^\dagger)^{a_2}_{c_2} \ldots (G^\dagger)^{a_p}_{c_p} G^{d_1}_{b_1} \ldots G^{d_q}_{b_q} \,. \tag{3.12}$$

Tensors can be combined into other tensors by
(a) *addition*

$$z^{ab\ldots c}_{d\ldots e} = \alpha x^{ab\ldots c}_{d\ldots e} + \beta y^{ab\ldots c}_{d\ldots e} \,, \qquad \alpha, \beta \in \mathbb{C} \,, \tag{3.13}$$

(b) *product*

$$z^{abcd}_{efg} = x^{abc}_e y^d_{fg} \,, \tag{3.14}$$

(c) *contraction:* Setting an upper and lower index equal and summing over all of its values yields a tensor $z \in V^{p-1} \otimes \bar{V}^{q-1}$ without these indices:

$$z^{bc\ldots d}_{e\ldots f} = x^{abc\ldots d}_{e\ldots af} \,, \qquad z^{ad}_e = x^{abc}_e y^d_{cb} \,. \tag{3.15}$$

A tensor $x \in V^p \otimes \bar{V}^q$ transforms linearly under the action of g, so it can be considered a vector in the $d = n^{p+q}$ dimensional vector space $\tilde{V}$. We can replace the array of its indices by one collective index:

$$x_\alpha = x^{a_1 a_2 \ldots a_q}_{b_1 \ldots b_p} . \tag{3.16}$$

One could be more explicit and give a table like

$$x_1 = x^{11\ldots 1}_{1\ldots 1}\,,\; x_2 = x^{21\ldots 1}_{1\ldots 1}, \ldots,\; x_d = x^{nn\ldots n}_{n\ldots n}\,, \tag{3.17}$$

but that is unnecessary, as we shall use the compact index notation only as a shorthand.

Definition. *Hermitian conjugation* is effected by complex conjugation and index transposition:

$$(h^\dagger)^{ab}_{cde} \equiv (h^{edc}_{ba})^* . \tag{3.18}$$

Complex conjugation interchanges upper and lower indices, as in (3.6); transposition reverses their order. A matrix is *hermitian* if its elements satisfy

$$(M^\dagger)^a_b = M^a_b . \tag{3.19}$$

Definition. The tensor conjugate to x_α has form

$$x^\alpha = x^{b_p \ldots b_1}_{a_q \ldots a_2 a_1} . \tag{3.20}$$

Combined, the above definitions lead to the hermitian conjugation rule for collective indices: a collective index is raised or lowered by interchanging the upper and lower indices and reversing their order:

$$_\alpha = \left\{ \begin{array}{c} a_1 a_2 \ldots a_q \\ b_1 \ldots b_p \end{array} \right\} \quad \leftrightarrow \quad ^\alpha = \left\{ \begin{array}{c} b_p \ldots b_1 \\ a_q \ldots a_2 a_1 \end{array} \right\} . \tag{3.21}$$

This transposition convention will be motivated further by the diagrammatic rules of sect. 4.1.

The tensor rep (3.12) can be treated as a $[d \times d]$ matrix

$$G_\alpha^\beta = G^{a_1 a_2 \ldots a_q}{}_{b_1 \ldots b_p}{}, {}^{d_p \ldots d_1}_{c_q \ldots c_2 c_1}\,, \tag{3.22}$$

and the tensor transformation (3.11) takes the usual matrix form

$$x'_\alpha = G_\alpha^\beta x_\beta . \tag{3.23}$$

3.2 INVARIANTS

Definition. The vector $q \in V$ is an *invariant vector* if for any transformation $g \in \mathcal{G}$

$$q = Gq . \tag{3.24}$$

Definition. A tensor $x \in V^p \otimes \bar{V}^q$ is an *invariant tensor* if for any $g \in G$

$$x^{a_1 a_2 \ldots a_p}_{b_1 \ldots b_q} = (G^\dagger)^{a_1}_{c_1} (G^\dagger)^{a_2}_{c_2} \ldots G^{d_1}_{b_1} \ldots G^{d_q}_{b_q} x^{c_1 c_2 \ldots c_p}_{d_1 \ldots d_q} . \tag{3.25}$$

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We can state this more compactly by using the notation of (3.22)

$$x_\alpha = G_\alpha^\beta x_\beta . \tag{3.26}$$

Here we treat the tensor $x^{a_1 a_2 \ldots a_p}_{b_1 \ldots b_q}$ as a vector in $[d \times d]$ dimensional space, $d = n^{p+q}$.

If a bilinear form $M(\bar{x}, y) = x^a M_a^b y_b$ is invariant for all $g \in \mathcal{G}$, the matrix

$$M_a^b = G_a^c (G^\dagger)^b_d M_c^d \tag{3.27}$$

is an *invariant matrix* . Multiplying with G_b^e and using the unitary condition (3.10), we find that the invariant matrices *commute* with all transformations $g \in \mathcal{G}$:

$$[G, M] = 0 . \tag{3.28}$$

If we wish to treat a tensor with equal number of upper and lower indices as a matrix $M : V^p \otimes \bar{V}^q \to V^p \otimes \bar{V}^q$,

$$M_\alpha^\beta = M^{a_1 a_2 \ldots a_q}{}_{b_1 \ldots b_p}{}, {}^{d_p \ldots d_1}_{c_q \ldots c_2 c_1}\,, \tag{3.29}$$

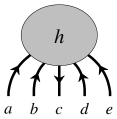
then the invariance condition (3.26) will take the commutator form (3.28).

Definition. We shall refer to an invariant relation between p vectors in V and q vectors in $\bar{V}$ which can be written as a homogeneous polynomial in terms of vector components, such as

$$H(x, y, \bar{z}, \bar{r}, \bar{s}) = h^{ab}_{cde} x_b y_a s^e r^d z^c\,, \tag{3.30}$$

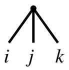
as an *invariant* in $V^q \otimes \bar{V}^p$ (repeated indices, as always, summed over). In this example, the coefficients h^{ab}_{cde} are components of invariant tensor $h \in V^3 \otimes \bar{V}^2$, obeying the invariance condition (3.25).

Diagrammatic representation of tensors, such as

$h^{ab}_{cd} =$  (3.31)

makes it easier to distinguish different types of invariant tensors. We shall explain in great detail our conventions for drawing tensors in sect. 4.1; sketching a few simple examples should suffice for the time being.

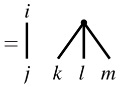
The standard example of a defining vector space is our 3-dimensional Euclidean space: $V = \bar{V}$ is the space of all 3-component real vectors $(n = 3)$, and examples of invariants are the length $L(x, x) = \delta_{ij} x_i x_j$ and the volume $V(x, y, z) = \epsilon_{ijk} x_i y_j z_k$. We draw the corresponding invariant tensors as

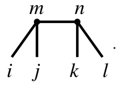
$\delta_{ij} = i$ $j\,,\qquad \epsilon_{ijk} =$  . (3.32)

Definition. A *composed* invariant tensor can be written as a product and/or contraction of invariant tensors.

Examples of composed invariant tensors are

$$\delta_{ij}\epsilon_{klm} = \qquad , \qquad \epsilon_{ijm}\delta_{mn}\epsilon_{nkl} = \qquad . \tag{3.33}$$





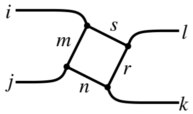
The first example corresponds to a product of the two invariants $L(x,y)V(z,r,s)$. The second involves an index *contraction*; we can write this as $V\left(x,y,\frac{d}{dz}\right)V(z,r,s)$.

In order to proceed, we need to distinguish the "primitive" invariant tensors from the infinity of composed invariants. We begin by defining a finite basis for invariant tensors in $V^p \otimes \bar{V}^q$:

Definition. A *tree invariant* can be represented diagrammatically as a product of invariant tensors involving no loops of index contractions. We shall denote by $T = \{\mathbf{t}_0, \mathbf{t}_1 \ldots \mathbf{t}_r\}$ a (maximal) set of r linearly independent tree invariants $\mathbf{t}_\alpha \in V^p \otimes \bar{V}^q$. As any linear combination of $\mathbf{t}_\alpha$ can serve as a basis, we clearly have a great deal of freedom in making informed choices for the basis tensors.

<u>*Example:*</u> Tensors (3.33) are tree invariants. The tensor

$$h_{ijkl} = \epsilon_{ims}\epsilon_{jnm}\epsilon_{krn}\epsilon_{\ell sr} = \qquad , \tag{3.34}$$

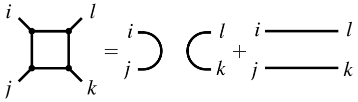


is not a tree invariant, as it involves a loop.

Definition. An invariant tensor is called a *primitive* invariant tensor, if it cannot be expressed as a combination of tree invariants composed from lower rank primitive invariant tensors. Let $P = \{p_1, p_2, \ldots p_k\}$ be the set of all primitives.

For example, the Kronecker delta and the Levi-Civita tensor (3.32) are the primitive invariant tensors of our 3-dimensional space. The loop contraction (3.34) is not a primitive, because by the Levi-Civita completeness relation (6.28) it reduces to a sum of tree contractions:

$$= \delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} \,, \tag{3.35}$$



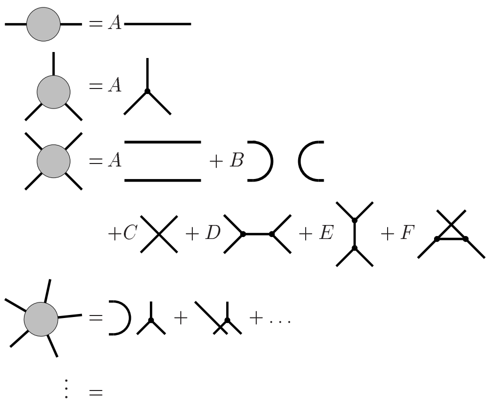
(the Levi-Civita tensor is discussed in sect. 6.3).

Primitiveness assumption. Any invariant tensor $h \in V^p \otimes \bar{V}^q$ can be expressed as a linear sum over the tree invariants $T \in V^q \otimes \bar{V}^p$

$$h = \sum_T h^\alpha \mathbf{t}_\alpha \,. \tag{3.36}$$

In contradistinction to arbitrary composite invariant tensors, the number of tree invariants for a fixed number of external indices is finite. For example, given the

$n = 3$ dimensions primitives $P = \{\delta_{ij}, f_{ijk}\}$, any invariant tensor $h \in V^p$ (here denoted by a blob) must be expressible as



3.2.1 Algebra of invariants

Any invariant tensor of matrix form (3.29)

$$M_\alpha^\beta = M_{b_1 \ldots b_p}^{a_1 a_2 \ldots a_q}\,{}_{,\,c_q \ldots c_2 c_1}^{\,d_p \ldots d_1}$$

which maps $V^q \otimes \bar{V}^p \to V^q \otimes \bar{V}^p$ can be expanded in the basis (3.36). The bases $\mathbf{t}_\alpha$ are themselves matrices in $V^q \otimes \bar{V}^p \to V^q \otimes \bar{V}^p$, and the matrix product of two basis elements is also an element of $V^q \otimes \bar{V}^p \to V^q \otimes \bar{V}^p$ and can be expanded in the minimal basis:

$$\mathbf{t}_\alpha \mathbf{t}_\beta = \sum_{\gamma \in T} (t_\alpha)_\beta{}^\gamma \mathbf{t}_\gamma \,. \tag{3.39}$$

As the number of tree invariants composed from the primitives is finite, under matrix multiplication the bases $\mathbf{t}_\alpha$ form a finite algebra, with the coefficients $(t_\alpha)_\beta{}^\gamma$ giving their multiplication table. The multiplication coefficients $(t_\alpha)_\beta{}^\gamma$ form a $[r \times r]$-dimensional matrix rep of $\mathbf{t}_\alpha$ acting on the vector $(\mathbf{e}, \mathbf{t}_1, \mathbf{t}_2, \cdots \mathbf{t}_{r-1})$. Given a basis, we can evaluate the matrices $\mathbf{e}_\beta{}^\gamma$, $(t_1)_\beta{}^\gamma$, $(t_2)_\beta{}^\gamma, \cdots (t_{r-1})_\beta{}^\gamma$ and their eigenvalues. For at least one of these matrices all eigenvalues will be distinct (or we have failed to chose a minimal basis). The projection operator technique of sect. 3.4 will enable us to exploit this fact to decompose the $V^q \otimes \bar{V}^p$ space into r irreducible subspaces.

This can be said in another way; the choice of basis $\{\mathbf{e}, \mathbf{t}_1, \mathbf{t}_2, \cdots \mathbf{t}_{r-1}\}$ is arbitrary, the only requirement being that the basis elements are linearly independent. Finding a $(t_\alpha)_\beta{}^\gamma$ with all eigenvalues distinct is all we need to construct an orthogonal basis $\{P_0, P_1, P_2, \cdots P_{r-1}\}$, where the basis matrices P_i are the projection operators, to be constructed below in sect. 3.4.

3.3 INVARIANCE GROUPS

So far we have defined invariant tensors as the tensors invariant under transformations of a given group. Now we proceed in the other direction: given a set of tensors, what is the group of transformations that leaves them invariant?

Given a full set of primitives (3.30) $P = \{p_1, p_2, \ldots, p_k\}$, meaning that *no other* primitives exist, we wish to determine all possible transformations that preserve this given set of invariant relations.

Definition. An *invariance group* $\mathcal{G}$ is the set of all linear transformations (3.25) which preserve the primitive invariant relations (and, by extension, *all* invariant relations)

$$\begin{aligned} p_1(x, \bar{y}) &= p_1(Gx, \bar{y}G^\dagger) \\ p_2(x, y, z, \ldots) &= p_2(Gx, Gy, Gz \ldots)\,, \qquad \ldots . \end{aligned} \tag{3.40}$$

Unitarity (3.10) guarantees that all contractions of primitive invariant tensors, and hence all composed tensors $h \in H$ are also invariant under action of $\mathcal{G}$. As we consider $\mathcal{G}$ is unitary, it follows from (3.10) that the list of primitives must always include the Kronecker delta.

Example 1. If $p^a q_a$ is an invariant of $\mathcal{G}$

$$p'^a q'_a = p^b (G^\dagger G)_b^c q_c = p^a q_a\,, \tag{3.41}$$

then $\mathcal{G}$ is the full *unitary group* $U(n)$ (invariance group of the complex norm $|x|^2 = x^b x_a \delta_b^a$), whose elements satisfy

$$G^\dagger G = 1\,. \tag{3.42}$$

Example 2. If we wish the z-direction to be invariant in our 3-dimensional space, $q = (0, 0, 1)$ is an invariant vector (3.24), and the invariance group is $O(2)$, the group of all rotations in the x-y plane.

3.3.0.1 Which rep is "defining"?

1. The defining space V *need not* carry the lowest dimensional rep of $\mathcal{G}$; it is merely the space in terms of which we chose to define the primitive invariants.

2. We shall always assume that the Kronecker delta δ_a^b is one of the primitive invariants, *ie.* that $\mathcal{G}$ is a unitary group whose elements satisfy (3.42). This restriction to unitary transformations is not essential, but it simplifies proofs of full reducibility. The results, however, apply as well to the finite-dimensional reps of non-compact groups, such as the Lorentz group $SO(3, 1)$.

3.4 PROJECTION OPERATORS

For M, a hermitian matrix, there exists a diagonalizing unitary matrix C such that:

$$CMC^\dagger = \begin{pmatrix} \begin{array}{cc|} \lambda_1 & 0 \\ 0 & \lambda_1 \\ \hline \end{array} & 0 & 0 \\ 0 & \boxed{\begin{array}{cccc} \lambda_2 & 0 & \ldots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \vdots \\ 0 & & \ldots & \lambda_2 \end{array}} & 0 \\ 0 & 0 & \begin{array}{|cc} \hline \lambda_3 & \ldots \\ \vdots & \ddots \end{array} \end{pmatrix}, \quad \lambda_i \neq \lambda_j\,. \tag{3.43}$$

Here λ_i are the r distinct roots of the minimal characteristic polynomial

$$\prod_{i=1}^{r} (M - \lambda_i \mathbf{1}) = 0\,, \tag{3.44}$$

(the characteristic equations will be discussed in sect. 6.6). In the matrix $C(M - \lambda_2 \mathbf{1})C^\dagger$ the eigenvalues corresponding to λ_2 are replaced by zeroes:

$$\begin{pmatrix} \begin{array}{ccc|} \lambda_1 - \lambda_2 & & \\ & \lambda_1 - \lambda_2 & \\ & & \lambda_1 - \lambda_2 \\ \hline \end{array} & & \\ & \boxed{\begin{array}{ccc} 0 & & \\ & \ddots & \\ & & 0 \end{array}} & \\ & & \begin{array}{|ccc} \hline \lambda_3 - \lambda_2 & & \\ & \lambda_3 - \lambda_2 & \\ & & \ddots \end{array} \end{pmatrix},$$

and so on, so the product over all factors $(M - \lambda_2 \mathbf{1})(M - \lambda_3 \mathbf{1}) \ldots$ with exception of the $(M - \lambda_1 \mathbf{1})$ factor has non-zero entries only in the subspace associated with λ_1:

$$C \prod_{j \neq 1} (M - \lambda_j \mathbf{1}) C^\dagger = \prod_{j \neq 1} (\lambda_1 - \lambda_j) \begin{pmatrix} \begin{array}{ccc|} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline \end{array} & 0 \\ 0 & \begin{array}{|cccc} \hline 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{array} \end{pmatrix}.$$

In this way, we can associate with each distinct root λ_i a *projection operator* P_i

$$P_i = \prod_{j \neq i} \frac{M - \lambda_j \mathbf{1}}{\lambda_i - \lambda_j}\,, \tag{3.45}$$

which is identity on the ith subspace, and zero elsewhere. For example, the projection operator onto the λ_1 subspace is

$$P_1 = C^\dagger \left(\begin{array}{ccc|cccc} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ \hline & & & 0 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & \ddots \end{array} \right) C \,. \tag{3.46}$$

The matrices P_i are *orthogonal*

$$P_i P_j = \delta_{ij} P_j \,, \qquad \text{(no sum on } j) \,, \tag{3.47}$$

and satisfy the *completeness relation*

$$\sum_{i=1}^{r} P_i = \mathbf{1} \,. \tag{3.48}$$

As $\mathrm{tr}\,(C P_i C^+) = \mathrm{tr}\, P_i$, the dimension of the ith subspace is given by

$$d_i = \mathrm{tr}\, P_i \,. \tag{3.49}$$

It follows from the characteristic equation (3.44) and the form of the projection operator (3.45) that λ_i is the eigenvalue of M on P_i subspace:

$$M P_i = \lambda_i P_i \,, \qquad \text{(no sum on } i) \,. \tag{3.50}$$

Hence, any matrix polynomial $f(M)$ takes the scalar value $f(\lambda_i)$ on the P_i subspace

$$f(M) P_i = f(\lambda_i) P_i \,. \tag{3.51}$$

This, of course, is the real reason why one wants to work with irreducible reps: they render matrices and "operators" harmless c-numbers.

3.5 FURTHER INVARIANTS

Suppose there exists several linearly independent invariant $[d \times d]$ hermitian matrices $M_1, M_2, \ldots$ and that we have used M_1 to decompose the d-dimensional vector space $\tilde{V} = \Sigma \oplus V_i$. Can M_2 be used to further decompose V_i? This is the standard problem of quantum mechanics (simultaneous observables), and the answer is that further decomposition is possible if, and only if, the invariant matrices commute,

$$[M_1, M_2] = 0 \,, \tag{3.52}$$

or, equivalently, if all projection operators commute

$$P_i P_j = P_j P_i \,. \tag{3.53}$$

Usually the simplest choices of independent invariant matrices do not commute. In that case, the projection operators P_i constructed from M_1 can be used to project commuting pieces of M_2:

$$M_2^{(i)} = P_i M_2 P_i \,, \qquad \text{(no sum on } i) \,.$$

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That $M_2^{(i)}$ commutes with M_1 follows from the orthogonality of P_i:

$$[M_2^{(i)}, M_1] = \sum_j \lambda_j [M_2^{(i)}, P_j] = 0 \,. \tag{3.54}$$

Now the characteristic equation for $M_2^{(i)}$ (if nontrivial) can be used to decompose V_i subspace.

An invariant matrix M induces a decomposition only if its diagonalized form (3.43) has more than one distinct eigenvalue; otherwise it is proportional to the unit matrix and commutes trivially with all group elements. A rep is said to be *irreducible*, if all invariant matrices that can be constructed are proportional to the unit matrix.

In particular, the primitiveness relation (3.37) is a statement that the defining rep is *assumed* irreducible.

According to (3.28), an invariant matrix M commutes with group transformations $[G, M] = 0$. Projection operators (3.45) constructed from M are polynomials in M, so they also commute with all $g \in \mathcal{G}$:

$$[G, P_i] = 0 \,, \tag{3.55}$$

(remember that P_i are also invariant $[d \times d]$ matrices). Hence, a $[d \times d]$ matrix rep can be written as a direct sum of $[d_i \times d_i]$ matrix reps

$$G = \mathbf{1} G \mathbf{1} = \sum_{i,j} P_i G P_j = \sum_i P_i G P_i = \sum_i G_i \,. \tag{3.56}$$

In the diagonalized rep (3.46), the matrix G has a block diagonal form:

$$C G C^\dagger = \begin{bmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0 \\ 0 & 0 & \ddots \end{bmatrix} \,, \qquad G = \sum_i C^i G_i C_i \,. \tag{3.57}$$

Representation G_i acts only on the d_i dimensional subspace V_i consisting of vectors $P_i q$, $q \in \tilde{V}$. In this way an invariant $[d \times d]$ hermitian matrix M with r distinct eigenvalues induces a decomposition of a d-dimensional vector space $\tilde{V}$ into a direct sum of d_i-dimensional vector subspaces V_i

$$\tilde{V} \overset{M}{\to} V_1 \oplus V_2 \oplus \ldots \oplus V_r \,. \tag{3.58}$$

For a more detailed discussion of recursive reduction, consult appendix A.

Chapter Four

Diagrammatic notation

The subject of this monograph is some aspects of the representation theory of Lie groups. However, it is not written in the conventional tensor notation but instead in terms of an equivalent diagrammatic notation. We shall refer to this style of carrying out group-theoretic calculations as birdtracks. The advantage of diagrammatic notation will become self-evident, we hope. Two of the principal benefits are that it eliminates "dummy indices", and that it does not force group-theoretic expressions into the 1-dimensional tensor format (both being means whereby identical tensor expressions can be made to look totally different).

4.1 BIRDTRACKS

We shall often find it convenient to represent agglomerations of invariant tensors by "birdtracks", a group-theoretical version of Feynman diagrams. Tensors will be represented by "vertices" and contractions by "propagators".

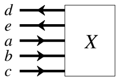
Diagrammatic notation has several advantages over the tensor notation. Diagrams do not require dummy indices, so explicit labeling of such indices is unnecessary. More to the point, for a human eye it is easier to identify topologically identical diagrams than to recognize equivalence between the corresponding tensor expressions.

In the birdtrack notation, the Kronecker delta is a "propagator":

$$\delta^a_b = b \longleftarrow a\,. \tag{4.1}$$

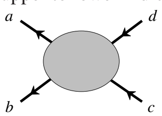
For a *real* defining space there is no distinction between V and $\bar{V}$, or up and down indices, and the lines do not carry arrows.

Any invariant tensor can be drawn as a generalized vertex:

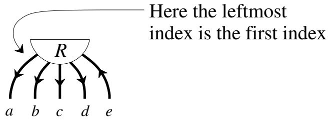
$$x_\alpha = x^{abc}_{de} =$$  . (4.2)

Whether the vertex is drawn as a box or a circle or a dot is matter of taste. The orientation of propagators and vertices in the plane of the drawing is likewise irrelevant. The only rules are

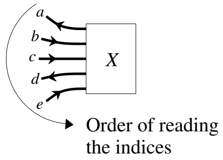
(1) Arrows point *away from the upper* indices and *toward the lower* indices; the line flow is "downward", from upper to lower indices:

$$h^{cd}_{ab} =$$  . (4.3)

(2) Diagrammatic notation must indicate which in (out) arrow corresponds to the first upper (lower) index of the tensor (unless the tensor is cyclically symmetric);

$$R^e_{abcd} =$$  . (4.4)

(3) The indices are read in the *counterclockwise* order around the vertex:

$$x^{bce}_{ad} =$$  . (4.5)

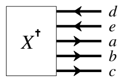
(The upper and the lower indices are read separately in the counterclockwise order; their relative ordering does not matter.)

In the examples of this section we index the external lines for reader's convenience, but indices can always be omitted. An internal line implies a summation over corresponding indices, and for external lines the equivalent points on each diagram represent the same index in all terms of a diagrammatic equation.

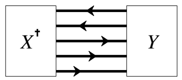
Hermitian conjugation (3.18) does two things:

(a) it exchanges the upper and the lower indices, *ie.* it reverses the directions of the arrows

(b) it reverses the order of the indices, *ie.* it transposes a diagram into its mirror image. For example $x^\dagger$, the tensor conjugate to (4.5), is drawn as

$$x^\alpha = x^{ed}_{cba} =$$  (4.6)

and a contraction of tensors $x^\dagger$ and y is drawn as

$$x^\alpha y_\alpha = x^{b_p \ldots b_1}_{a_q \ldots a_2 a_1} y^{a_1 a_2 \ldots a_q}_{b_1 \ldots b_p} =$$  . (4.7)

4.2 CLEBSCH-GORDAN COEFFICIENTS

Consider the product

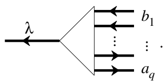
$$\begin{pmatrix} \begin{array}{cc|} 0 & \\ & 0 \\ \hline \end{array} & & \\ & \boxed{\begin{array}{ccc} 1 & & \\ & 1 & \\ & & 1 \end{array}} & \\ & & \begin{array}{|cccc} \hline 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{array} \end{pmatrix} C \tag{4.8}$$

of the two terms in the diagonal representation of a projection operator (3.46). This matrix has non-zero entries only in the d_λ rows of subspace V_λ. We collect them in a $[d_\lambda \times d]$ rectangular matrix $(C_\lambda)^\alpha_\sigma$, $\alpha = 1, 2, \ldots d$, $\sigma = 1, 2, \ldots d_\lambda$:

$$C_\lambda = \underbrace{\left.\begin{pmatrix} (C_\lambda)^1_1 & \ldots & (C_\lambda)^d_1 \\ \vdots & & \vdots \\ & & (C_\lambda)^d_{d_\lambda} \end{pmatrix}\right\}}_{d} d_\lambda \,. \tag{4.9}$$

The index α in $(C_\lambda)^\alpha_\sigma$ stands for all tensor indices associated with the $d = n^{p+q}$ dimensional tensor space $V^p \otimes \bar{V}^q$. In the birdtrack notation these indices are explicit:

$$(C_\lambda)_{\sigma,}{}^{b_p \ldots b_1}_{a_q \ldots a_2 a_1} = \tag{4.10}$$



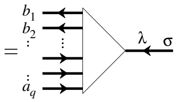
Such rectangular arrays are called *Clebsch-Gordan coefficients* (hereafter referred to as "clebsches" for short). They are explicit mappings $\tilde{V} \to V_\lambda$. The conjugate mapping $V_\lambda \to \tilde{V}$ is provided by the product

$$C^\dagger \begin{pmatrix} \begin{array}{cc|} 0 & \\ & 0 \\ \hline \end{array} & & \\ & \boxed{\begin{array}{ccc} 1 & & \\ & 1 & \\ & & 1 \end{array}} & \\ & & \begin{array}{|cccc} \hline 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{array} \end{pmatrix} , \tag{4.11}$$

which defines the $[d \times d_\lambda]$ rectangular matrix $(C^\lambda)^\sigma_\alpha$, $\alpha = 1, 2, \ldots d$, $\sigma = 1, 2, \ldots d_\lambda$:

$$C^\lambda = \underbrace{\left.\begin{pmatrix} (C^\lambda)^1_1 & \ldots & (C^\lambda)^{d_\lambda}_1 \\ \vdots & & \vdots \\ & & (C^\lambda)^{d_\lambda}_d \end{pmatrix}\right\}}_{d_\lambda} d$$

$$(C^\lambda)^{a_1 a_2 \ldots a_q}_{b_1 \ldots b_p},{}^{\sigma} = \tag{4.12}$$



The two rectangular Clebsch-Gordan matrices C^λ and C_λ are related by hermitian conjugation.

The tensors, we have considered in sect. 3.6, transform as tensor products of the defining rep (3.11). In general, tensors transform as tensor products of various reps, with indices running over the corresponding rep dimensions:

$$x^{a_{p+1} \ldots a_{p+q}}_{a_1 a_2 \ldots a_p} \qquad \text{where} \begin{array}{l} a_1 = 1, 2, \ldots, d_1 \\ a_2 = 1, 2, \ldots, d_2 \\ \vdots \\ a_{p+q} = 1, 2, \ldots, d_{p+q} \,. \end{array} \tag{4.13}$$

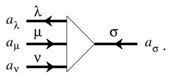
(4.14)

The action of transformation g on the index a_k is given by the $[d_k \times d_k]$ matrix rep G_k.

The Clebsch-Gordan coefficients are notoriously index-overpopulated, as they require a rep label and a tensor index for each rep in the tensor product. Diagrammatic notation alleviates this index plague in either of two ways:

(i) one can indicate a rep label on each line:

$$C^{a_\mu a_\nu}_{a_\lambda},{}^{a_\sigma} = \tag{4.15}$$



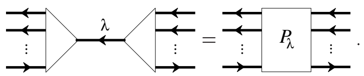
(an index, if written, is written at the end of a line; a rep label is written above the line);

(ii) one can draw the propagators (Kronecker deltas) for different reps with different kinds of lines. For example, we shall usually draw the adjoint rep with a thin line.

By the definition of clebsches (3.46), the λ rep projection operator can be written out in terms of Clebsch-Gordan matrices: $C^\lambda C_\lambda$:

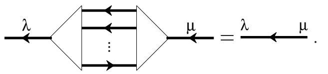
$$C^\lambda C_\lambda = P_\lambda \,, \qquad \text{(no sum on } i\text{)}$$

$$(C^\lambda)^{a_1 a_2 \ldots a_p}_{b_1 \ldots b_q},{}^{\alpha} (C_\lambda)_{\alpha,}{}^{d_q \ldots d_1}_{c_p \ldots c_2 c_1} = (P_\lambda)^{a_1 a_2 \ldots d_p}_{b_1 \ldots b_q},{}^{d_q \ldots d_1}_{c_p \ldots c_2 c_1} \tag{4.16}$$



A specific choice of clebsches is quite arbitrary. All relevant properties of projection operators (orthogonality, completeness, dimensionality) are independent of the explicit form of the diagonalization transformation C. Any set of C_λ is acceptable, as long as it satisfies the orthogonality and completeness conditions. From (4.8) and (4.11) it follows that C_λ are *orthogonal*:

$$C_\lambda C^\mu = \delta^\mu_\lambda \mathbf{1}\,,$$
$$(C_\lambda)_{\beta,}{}^{a_1 a_2 \ldots a_p}_{b_1 \ldots b_q} (C^\mu)_{a_p \ldots a_2 a_1}{}^{b_q \ldots b_1}{}_{,}{}^{\alpha} = \delta^\alpha_\beta \delta^\mu_\lambda$$

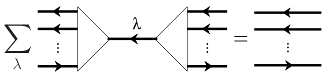


(4.17)

Here $\mathbf{1}$ is the $[d_\lambda \times d_\lambda]$ unit matrix, and C_λ's are multiplied as $[d_\lambda \times d]$ rectangular matrices.

The *completeness relation* (3.48)

$$\sum_\lambda C^\lambda C_\lambda = \mathbf{1}\,, \qquad ([d \times d] \text{ unit matrix})\,,$$
$$\sum_\lambda (C^\lambda)^{a_1 a_2 \ldots a_p}_{b_1 \ldots b_q}{}_{,}{}^{\alpha} (C_\lambda)_{\alpha,}{}^{d_q \ldots d_1}_{c_p \ldots c_2 c_1} = \delta^{a_1}_{c_1} \delta^{a_2}_{c_2} \ldots \delta^{d_q}_{b_q}$$



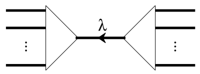
(4.18)

$$C^\lambda P_\mu = \delta^\mu_\lambda C^\lambda\,,$$
$$P_\lambda C^\mu = \delta^\mu_\lambda C^\mu \qquad (\text{no sum on } \lambda, \mu)\,, \tag{4.19}$$

follows immediately from (3.47) and (4.17).

4.3 ZERO- AND ONE-DIMENSIONAL SUBSPACES

If a projection operator projects onto a zero-dimensional subspace, it must vanish identically

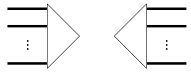
$d_\lambda = 0 \qquad \Rightarrow \qquad P_\lambda =$  $= 0\,.$ (4.20)

This follows from (3.46); d_λ is the number of 1's on the diagonal on the right-hand side. For $d_\lambda = 0$ the right-hand side vanishes. The general form of P_λ is

$$P_\lambda = \sum_{k=1}^{r} c_k M_k\,, \tag{4.21}$$

where M_k are the invariant matrices used in construction of the projector operators, and c_k are numerical coefficients. Vanishing of P_λ therefore implies a relation among invariant matrices M_k.

If a projection operator projects onto a 1-dimensional subspace, its expression, in terms of the Clebsch-Gordan coefficients (4.16), involves no summation, so we can omit the intermediate line

$d_\lambda = 1 \qquad \Rightarrow \qquad P_\lambda =$  $= (C^\lambda)^{a_1 a_2 \ldots a_p}_{b_1 \ldots b_q} (C_\lambda)^{d_q \ldots d_1}_{c_p \ldots c_2 c_1}\,.$

(4.22)

For any subgroup of $SU(n)$, the reps are unitary, with unit determinant. On the 1-dimensional spaces, the group acts trivially, $G = 1$. Hence, if $d_\lambda = 1$, the Clebsch-Gordan coefficient C_λ in (4.22) is an invariant tensor in $V^p \otimes \bar{V}^q$.

4.4 INFINITESIMAL TRANSFORMATIONS

A unitary transformation G, which is infinitesimally close to unity, can be written as

$$G_a^b = \delta_a^b + i D_a^b\,, \tag{4.23}$$

where D is a hermitian matrix with small elements, $|D_a^b| \ll 1$. The action of $g \in \mathcal{G}$ on the conjugate space is given by

$$(G^\dagger)_b^a = \delta_b^a - i D_b^a\,. \tag{4.24}$$

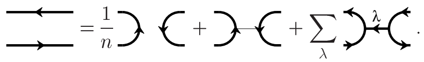
D can be parametrized by $N \leq n^2$ real parameters. N, the maximal number of independent parameters, is called the dimension of the group (also the dimension of the Lie algebra, or the dimension of the adjoint rep).

We shall consider only infinitesimal transformations, of form $G = 1 + iD$, $|D_b^a| \ll 1$. We do not study the entire group of invariances, but only the transformations (3.8) connected to the identity. For example, we shall not consider invariances under coordinate reflections.

The generators of infinitesimal transformations (4.23) are hermitian matrices and belong to the $D_b^a \in V \otimes \bar{V}$ space. However, not any element of $V \otimes \bar{V}$ generates an allowed transformation; indeed, one of the main objectives of group theory is to define the class of allowed transformations.

In sect. 3.4 we have described the general decomposition of a tensor space into (ir)reducible subspaces. As a particular case, consider the decomposition of $V \otimes \bar{V}$. The corresponding projection operators satisfy the completeness relation (4.18)

$$\mathbf{1} = \frac{1}{n} T + P_A + \sum_{\lambda \neq A} P_\lambda$$
$$\delta_d^a \delta_b^c = \frac{1}{n} \delta_b^a \delta_d^c + (P_A)_b^a{}_{,}{}^c_d + \sum_{\lambda \neq A} (P_\lambda)_b^a{}_{,}{}^c_d$$



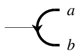
(4.25)

If δ^μ_λ is the only primitive invariant tensor, then $V \otimes \bar{V}$ decomposes into 2 subspaces, and there are no other irreducible reps. However, if there are further primitive invariant tensors, $V \otimes \bar{V}$ decomposes into more irreducible reps and therefore the sum over λ. Examples will abound in what follows. The singlet projection operator T/n always figures in this expansion, as $\delta^a_b, {}^c_d$ is always one of the invariant matrices (see the example worked out in sect. 2.2). Furthermore, the infinitesimal generators D^a_b must belong to at least one of the irreducible subspaces of $V \otimes \bar{V}$.

This subspace is called the *adjoint* space, and its special role warrants introduction of special notation. We shall refer to this vector space by letter A, in distinction to the defining space V of (3.6). We shall denote its dimension by N, label its tensor indices by $i, j, k \ldots$, denote the corresponding Kronecker delta by a thin, straight line

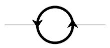
$$\delta_{ij} = i \;\text{———}\; j\,, \; i, j = 1, 2, \ldots, N\,, \tag{4.26}$$

and the corresponding Clebsch-Gordan coefficients by

$$(C_A)_{i,}{}^a_b = \frac{1}{\sqrt{a}}(T_i)^a_b = i$$  $$a, b = 1, 2, \ldots, n \qquad i = 1, 2, \ldots, N\,.$$

Matrices T_i are called the *generators* of infinitesimal transformations. Here a is an (uninteresting) overall normalization fixed by the orthogonality condition (4.17)

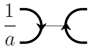
$$(T_i)^a_b(T_j)^b_a = \mathrm{tr}\,(T_iT_j) = a\,\delta_{ij}$$

 $= a$ ——— . (4.27)

The scale of T_i is not set, as any overall rescaling can be absorbed into the normalization a. For our purposes it will be most convenient to use $a = 1$ as the normalization convention. Other normalizations are commonplace. For example, $SU(2)$ Pauli matrices $T_i = \frac{1}{2}\sigma_i$ and $SU(n)$ Gell-Mann [73] matrices $T_i = \frac{1}{2}\lambda_i$ are conventionally normalized by fixing $a = 1/2$:

$$\mathrm{tr}\,(T_iT_j) = \frac{1}{2}\delta_{ij}\,. \tag{4.28}$$

The projector relation (4.16) expresses the adjoint rep projection operators in terms of the generators:

$$(P_A)^a_b, {}^c_d = \frac{1}{a}(T_i)^a_b(T_i)^c_d = \frac{1}{a}$$  . (4.29)

Clearly, the adjoint subspace is always included in the sum (4.25) (there must exist some allowed infinitesimal generators D^b_a, or otherwise there is no group to describe), but how do we determine the corresponding projection operator?

The adjoint projection operator is singled out by the requirement, that the group transformations do not affect the invariant quantities. (Remember, the group is *defined* as the totality of all transformations that leave the invariants invariant.) For every invariant tensor q, the infinitesimal group elements $G = 1 + iD$ must satisfy the invariance condition (3.24). Parametrizing D as a projection of an arbitrary hermitian matrix $H \in V \otimes \bar{V}$ into the adjoint space, $D = P_AH \in V \otimes \bar{V}$:

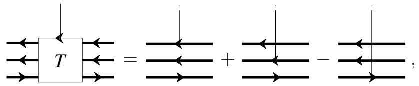
$$D^a_b = \frac{1}{a}(T_i)^a_b\epsilon_i\,, \qquad \epsilon_i = \frac{1}{a}\mathrm{tr}\,(T_iH)\,, \tag{4.30}$$

we obtain the *invariance condition*, which the *generators* must satisfy: they *annihilate* invariant tensors

$$T_iq = 0\,. \tag{4.31}$$

To state the invariance condition for an arbitrary invariant tensor, we need to define the generators in the tensor reps. By substituting $G = 1 + i\epsilon \cdot T + O(\epsilon^2)$ into (3.12) and keeping only the terms linear in ϵ, we find that the generators of infinitesimal transformations for tensor reps act by touching one index at a time:

$$\begin{aligned}(T_i)^{a_1a_2\ldots a_p}_{b_1\ldots b_q}, {}^{d_q\ldots d_1}_{c_p\ldots c_2c_1} &= (T_i)^{a_1}_{c_1}\delta^{a_2}_{c_2}\ldots\delta^{a_p}_{c_p}\delta^{d_1}_{b_1}\ldots\delta^{d_q}_{b_q}\\ &+\delta^{a_1}_{c_1}(T_i)^{a_2}_{c_2}\ldots\delta^{a_p}_{c_p}\delta^{d_1}_{b_1}\ldots\delta^{d_q}_{b_q} + \ldots + \delta^{a_1}_{c_1}\delta^{a_2}_{c_2}\ldots(T_i)^{a_p}_{c_p}\delta^{d_1}_{b_1}\ldots\delta^{d_q}_{b_q}\\ &-\delta^{a_1}_{c_1}\delta^{a_2}_{c_2}\ldots\delta^{a_p}_{c_p}(T_i)^{d_1}_{b_1}\ldots\delta^{d_q}_{b_q} - \ldots - \delta^{a_1}_{c_1}\delta^{a_2}_{c_2}\ldots\delta^{a_p}_{c_p}\delta^{d_1}_{b_1}\ldots(T_i)^{d_q}_{b_q}\,.\end{aligned} \tag{4.32}$$

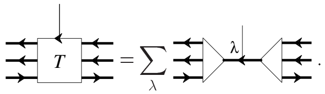


(4.33)

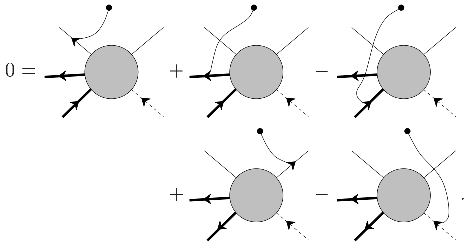
(with a relative minus sign between lines flowing in opposite directions). In other words, the Leibnitz rule obscured by a forest of indices.

Tensor reps of the generators decompose in the same way as the group reps (3.57)

$$T_i = \sum_\lambda C^\lambda T_i^{(\lambda)}C_\lambda\,. \tag{4.34}$$



The invariance conditions take a particularly suggestive form in the diagrammatic notation. (4.31) amounts to insertion of a generator into all external legs of the diagram corresponding to the invariant tensor q:


(4.35)

The insertions on the lines going into the diagram carry a minus sign relative to the insertions on the outgoing lines.

Clebsch-Gordan coefficients are also invariant tensors. Multiplying both sides of (3.57) with C_λ and using orthogonality (4.17), we obtain

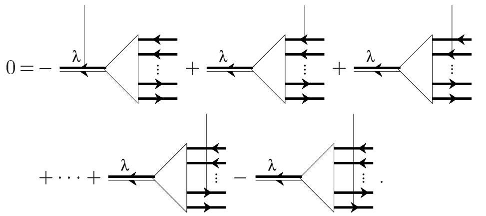
$$C_\lambda G = G_\lambda C_\lambda\,, \qquad (\text{no sum on } \lambda)\,. \tag{4.36}$$

The Clebsch-Gordan matrix C_λ is a rectangular $[d_\lambda \times d]$ matrix, hence $g \in \mathcal{G}$ acts on it with a $[d_\lambda \times d_\lambda]$ rep from the left, and a $[d \times d]$ rep from the right. (3.45) is the statement of invariance for rectangular matrices, analogous to (3.27), the statement of invariance for square matrices:

$$C_\lambda = G_\lambda^\dagger C_\lambda G\,, \qquad C^\lambda = G^\dagger C^\lambda G_\lambda\,. \tag{4.37}$$

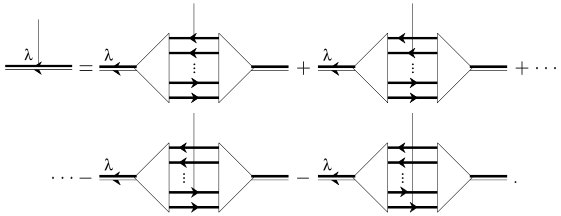
The invariance condition for the Clebsch-Gordan coefficients is a special case of (4.35), the invariance condition for any invariant tensor:

$$0 = -T_i^{(\lambda)} C_\lambda + C_\lambda T_i$$



(4.38)

The orthogonality condition (4.17) now yields the generators in λ rep in terms of the defining rep generators



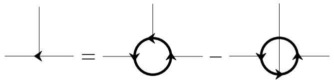
(4.39)

The reality of the adjoint rep. For hermitian generators, the adjoint rep is real, and the upper and lower indices need not be distinguished; the "propagator" needs no arrow. For non-hermitian choices of generators, the adjoint rep is complex ("gluon" lines carry arrows), but A and $\bar{A}$ are equivalent, as indices can be raised an lowered by the Cartan-Killing form $g_{ij} = \mathrm{tr}\,(T_i^\dagger T_j)$. The Cartan canonical basis $D = \epsilon_i H_i + \epsilon_\alpha E_\alpha + \epsilon_\alpha^* E_{-\alpha}$ is an example of a non-hermitian choice. Here we shall always assume that T_i are chosen hermitian.

4.5 LIE ALGEBRA

As the simplest example of computation of the generators of infinitesimal transformations acting on spaces other than the defining space, consider the adjoint rep. Using (4.39) on the $V \otimes \bar{V} \to A$ adjoint rep Clebsch-Gordan coefficients (*ie.*,

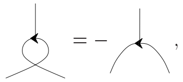
generators T_i), we obtain



(4.40)

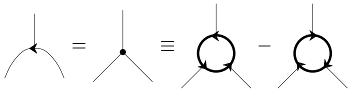
$$(T_i)_{jk} = (T_i)_a^c (T_k)_c^b (T_j)_b^a - (T_i)_a^c (T_j)_c^b (T_k)_b^a\,.$$

Our convention is always to assume that the generators T_i have been chosen hermitian. That means that ϵ_i in the expansion (4.30) is real; A is a real vector space, there is no distinction between upper and lower indices, and there is no need for arrows on the adjoint rep lines (4.26). However, the arrow on the adjoint rep generator (4.40) is necessary to define correctly the overall sign. If we interchange the two legs, the right-hand side changes sign



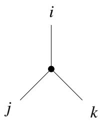
(4.41)

(the generators for real reps are always antisymmetric). This arrow has no absolute meaning; its direction is *defined* by (4.40). Actually, as the right-hand side of (4.40) is antisymmetric under interchange of any two legs, it is convenient to replace the arrow in the vertex by a more symmetric symbol, such as a dot:

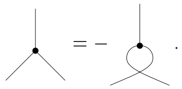


$$(T_i)_{jk} \equiv -iC_{ijk} = -\mathrm{tr}\,[T_i, T_j]T_k\,, \tag{4.42}$$

and replace the adjoint rep generators $(T_i)_{jk}$ by the fully antisymmetric structure constants iC_{ijk}. The factor i ensures their reality (in the case of hermitian generators T_i), and we keep track of the overall signs by always reading indices *counterclockwise* around a vertex

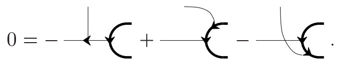
$-iC_{ijk} =$ 

(4.43)

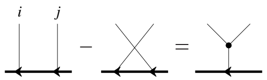


(4.44)

As all other clebsches, the generators must satisfy the invariance conditions (4.38):

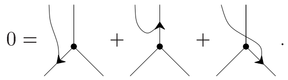


Redrawing this a little and replacing the adjoint rep generators (4.42) by the structure constants, we find that the generators obey the *Lie algebra* commutation relation

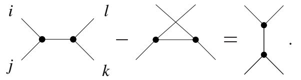


$$T_i T_j - T_j T_i = i C_{ijk} T_k \,. \tag{4.45}$$

In other words, the Lie algebra is simply a statement that T_i, the generators of invariance transformations, are themselves invariant tensors. The invariance condition for structure constants C_{ijk} is likewise



Rewriting this with the dot-vertex (4.42), we obtain



(4.46)

This is the Lie algebra commutator for the adjoint rep generators, known as the *Jacobi relation* for the structure constants

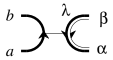
$$C_{ijm} C_{mkl} - C_{ljm} C_{mki} = C_{iml} C_{jkm} \,. \tag{4.47}$$

Hence, the Jacobi relation is also an invariance statement, this time the statement that the structure constants are invariant tensors.

Sign convention for C_{ijk}. A word of caution about using (4.45): vertex C_{ijk} is an oriented vertex. If the arrows are reversed (matrices T_i, T_j multiplied in reverse order), the right-hand side gets an overall minus sign.

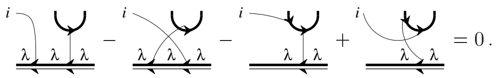
4.6 OTHER FORMS OF LIE ALGEBRA COMMUTATORS

Note that in our calculations we never need explicit generators; we use instead the projection operators for the adjoint rep. For rep λ they have the form

$(P_A)_b^a{}_\alpha^\beta =$  $\quad a, b = 1, 2, \ldots, n$

$$\alpha, \beta = 1, \ldots, d_\lambda \,. \tag{4.48}$$

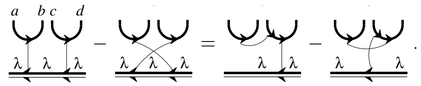
The invariance condition for a projection operator is



(4.49)

Contracting with $(T_i)_b^a$ and defining $[d_\lambda \times d_\lambda]$ matrices $(T_b^a)_\alpha^\beta \equiv (P_A)_b^a{}_\alpha^\beta$, we obtain

$$[T_b^a, T_d^c] = (P_A)_b^a{}_e^c T_d^e - T_e^c (P_A)_b^a{}_d^e$$



(4.50)

This is a common way of stating the Lie algebra conditions for the generators in an arbitrary rep λ. For example, for $U(n)$ the adjoint projection operator is simply a unit matrix (any hermitian matrix is a generator of unitary transformation, cf. chapter 9), and the right-hand side of (4.50) is given by

$$U(n), SU(n): \qquad [T_b^a, T_d^c] = \delta_b^c T_d^a - T_b^c \delta_d^a \,. \tag{4.51}$$

Another example is given by the orthogonal groups. The generators of rotations are antisymmetric matrices, and the adjoint projection operator antisymmetrizes generator indices:

$$SO(n): \qquad [T_{ab}, T_{cd}] = \frac{1}{2} \left\{ \begin{array}{c} g_{ac} T_{bd} - g_{ad} T_{bc} \\ -g_{bc} T_{ad} + g_{bd} T_{ac} \end{array} \right\} . \tag{4.52}$$

Apart from the normalization convention, these are the familiar Lorentz group commutation relations (we shall return to this in chapter 10).

4.7 IRRELEVANCY OF CLEBSCHES

As was emphasized in sect. 4.2, an explicit choice of clebsches is highly arbitrary; it corresponds to a particular coordinatization of the d_λ-dimensional subspace V_λ. For computational purposes clebsches are largely irrelevant. Nothing that a physicist wants to compute depends on an explicit coordinatization. For example, in QCD the physically interesting objects are color singlets, and all color indices are summed over: one needs only an expression for the projection operators (4.29), not for the C_λ's separately.

Again, a nice example is the Lie algebra generators T_i. Explicit matrices are often constructed (Gell-Mann λ_i matrices, Cartan's canonical weights); however, in any singlet they always appear summed over the adjoint rep indices, as in (4.29). The summed combination of clebsches is just the adjoint rep projection operator, a very simple object compared with explicit T_i matrices (P_A is typically a combination of a few Kronecker deltas), and much simpler to use in explicit evaluations. As we shall show by many examples, all rep dimensions, casimirs, *etc*., are computable once the projection operators for the reps involved are known. Explicit clebsches are superfluous from the computational point of view; we use them chiefly to state general theorems without recourse to any explicit realizations.

However, if one has to compute non-invariant quantities, such as subgroup embeddings, explicit clebsches might be very useful. Gell-Mann [73] invented λ_i matrices in order to embed $SU(2)$ of isospin into $SU(3)$ of the eightfold way. Cartan's canonical form for generators, summarized by Dynkin labels of a rep, table 7.7, is a very powerful tool in the study of symmetry breaking chains [145]. The same can be achieved with decomposition by invariant matrices (a nonvanishing expectation value for a direction in the defining space defines the little group of transformations in the remaining directions), but the tensorial technology in this context is underdeveloped compared to the canonical methods.

4.8 A BRIEF HISTORY OF BIRDTRACKS

In this monograph well developed conventional subjects - symmetric group, Lie algebras (and, to a lesser extent, continuous Lie groups) - are presented in a somewhat unconventional way, in a flavor of diagrammatic notation that I refer to as “birdtracks". Similar diagrammatic notations have been invented many times before, and continue to be invented within new research areas.

It is quite likely that since Sophus Lie’s days many have doodled birtracks in private without publishing them, partially out of sense of gravitas and no insignificant part because preparing these doodles for publications is even today a painful thing. I have seen unpublished 1960’s course notes of J.G. Belinfante [2], very much like the birtracks drawn here, and there are surely many other such doodles lost in the mists of time.

The methods used here come down to us along two distinct lineages, one that that can be traced to Wigner, and the other to Feynman.

Wigner’s 1930’s theory, elegantly presented in his group theory monograph [158], is still the best book on what physics is to be extracted from symmetries, be it atomic, nuclear, statistical, many-body or particle physics: all physical predictions (“spectroscopic levels”) are expressed in terms of Wigner’s $3n$-j coefficients, which can be evaluated by means of recursive or combinatorial algorithms. As explained here in chapter 5, decomposition (5.8) of tensor products into irreducible reps implies that any invariant number characterizing physical system with a given symmetry corresponds to one or several “vacuum bubbles”, trivalent graphs (a graph in which every vertex joins three links), such as those listed in table 5.1.

Since 1930’s much of the group-theoretical work on atomic and nuclear physics had focused on explicit construction of Clebsch-Gordan coefficients for the rotation group - $SO(2)$, $SU(2)$. The first paper recasting Wigner theory in graphical form appears to be a 1956 paper by I.B. Levinson [104], further developed in the influental 1960 monograph by A. P. Yutsis (later A. Jucys), I. Levinson and V. Vanagas [165], published in English in 1962 (see also refs. [59, 14]). The most recent contribution to this tradition, a very stimulating book by G. E. Stedman [149] covers a broad range of applications, including the methods introduced in the first version of present monograph [45].

The main drawback of such diagrammatic notations is lack of standardization, especially in the case of Clebsch-Gordan coefficients. In addition, the diagrammatic notations designed for atomic and nuclear spectroscopy are complicated by various phase conventions.

If diagrammatic notation is to succeed, it need be not only precise, but also beautiful. It is in this sense that this monograph belongs to the tradition of R.P. Feynman, whose sketches of the very first “Feynman diagrams” in his fundamental 1949 Q.E.D. paper [66] are beautiful to behold. Similarly, R. Penrose’s [132, 133] way of drawing symmetrizers and antisymmetrizers, adopted here in chapter 6, is imbued with a very Penrose aestethics, and even though the book is in black and white, one knows that he had drawn them in color. (Penrose, however, credits Aitken [5] with introducing this notation in 1939).

In introducing birtrack notation in 1975 I was inspired by “Feynman diagrams”

and the elegance of Penrose’s binors [132]. I liked G. ’t Hooft [86] 1974 double-line notation for $U(n)$ gluon group-theory weights, and have introduced analogous notation for $SU(n)$, $SO(n)$ and $Sp(n)$ in my 1976 paper [36]. The challenge was to do the same for the exceptional Lie algebras, and I suceeded [36], except for E_8 which came later.

In quantum groups literature graphs composed of vertices (4.42) are called trivalent. The Jacobi relation (4.46) in diagrammatic form was published [36] in 1976; though it seems surprising, I have not seen it in earlier literature. This set of diagrams has since been given moniker IHX by D. Bar-Natan [8]. It’s invention has been credited by N. Habegger [81] to Morita [196, 197], and by A.M. Cohen and R. de Man [29] to El Houari [60] (who, in turn, does refer to ref. [36]). Somewhat more mysteriously, in recent literature the birdtracks version of the Lie algebra commutator (4.45) appears to go under pseudonym "the STU relation".

So why call this “birdtracks” and not “Feynman diagrams”? The difference is that here diagrams are not a memonic device, an aid in writing donw an integral that is to be evaluated by other techniques. In our applications, explicit construction of clebsches would be superfluous, and we need no phase conventions. Here “birdtracks” are everything - unlike “Feynman diagrams”, here all calculations are carried out in terms of birtracks, from start to finish. Left behind are blackboards and pages of squiggles of kind that made Bernice Durand exclaim: “What are these birdtracks!?” and thus give them the name.

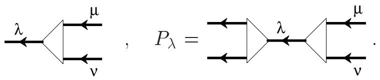
Chapter Five

Recouplings

Clebsches discussed in sect. 4.2 project a tensor in $V^p \otimes \bar{V}^q$ onto a subspace λ. In practice one usually reduces a tensor step by step, decomposing a 2-particle state at each step. While there is some arbitrariness in the order in which these reductions are carried out, the final result is invariant and highly elegant: any group-theoretical invariant quantity can be expressed in terms of Wigner 6-j coefficients.

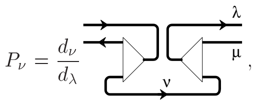
5.1 COUPLINGS AND RECOUPLINGS

We denote the clebsches for $\mu \otimes \nu \to \lambda$ by



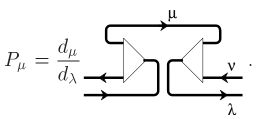
(5.1)

Here λ, μ, ν are rep labels, and the corresponding tensor indices are suppressed. Furthermore, if μ and ν are irreducible reps, the same clebsches can be used to project $\mu \otimes \bar{\lambda} \to \bar{\nu}$



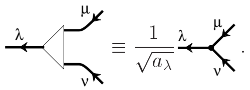
(5.2)

and $\nu \otimes \bar{\lambda} \to \bar{\mu}$



(5.3)

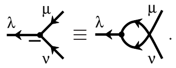
Here the normalization factors come from $P^2 = P$ condition. In order to draw the projection operators in a more symmetric way, we replace clebsches by 3-vertices:



(5.4)

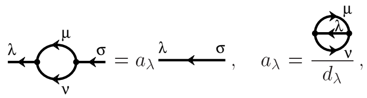
In this definition one has to keep track of the ordering of the lines around the vertex. If in some context the birdtracks look better with two legs interchanged, one can

use Yutsis' notation [165]



(5.5)

While all sensible clebsches are normalized by the orthonormality relation (4.17), in practice no two authors ever use the same normalization for 3-vertices (in other guises known as 3-j coefficients, Gell-Mann λ matrices, Cartan roots, Dirac γ matrices, *etc*, *etc*). For this reason we shall usually not fix the normalization

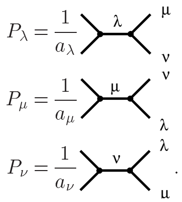


(5.6)

leaving the reader the option of substituting his favorite choice (such as $a = \frac{1}{2}$ if the 3-vertex stands for Gell-Mann $\frac{1}{2}\lambda_i$, *etc*).

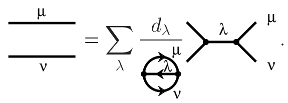
To streamline the discussion, we shall drop the arrows and most of the rep labels in the remainder of this chapter - they can always easily be reinstated.

The above three projection operators now take a more symmetric form:



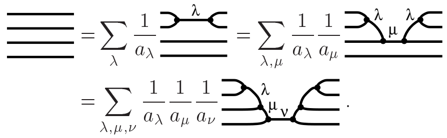
(5.7)

In terms of 3-vertices, the completeness relation (4.16) is



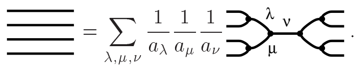
(5.8)

Any tensor can be decomposed by successive applications of the completeness relation:



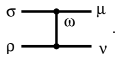
(5.9)

Hence, if we know clebsches for $\lambda \otimes \mu \to \nu$, we can also construct clebsches for $\lambda \otimes \mu \otimes \nu \otimes \ldots \to \rho$. However, there is no unique way of building up the clebsches; the above state can equally well be reduced by a different coupling scheme

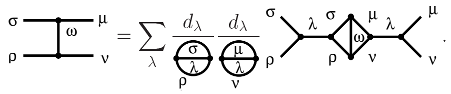


(5.10)

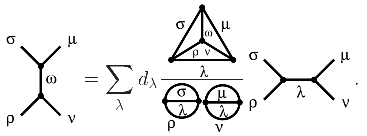
Consider now a process in which a particle in the rep μ interacts with a particle in the rep ν by exchanging a particle in the rep ω:


(5.11)

The final particles are in reps ρ and σ. To evaluate the contribution of this exchange to the spectroscopic levels of the $\mu-\nu$ particles system, we insert the Clebsch-Gordan series

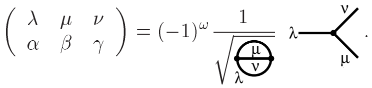

(5.12)

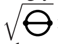
By assumption λ is irreducible, so we have a recoupling relation between the exchanges in "s" and "t channels":

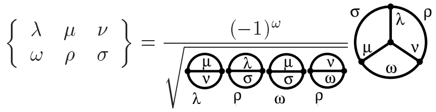

(5.13)

We shall refer to  as 3-j coefficients and  as 6-j coefficients, committing ourselves to no particular normalization convention.

In atomic physics it is customary to absorb  into the 3-vertex and define a 3-j symbol [135, 158]


(5.14)

Here $\alpha = 1, 2, \ldots, d_\lambda$, *etc*, are indices, λ, μ, ν rep labels, and ω the phase convention. Fixing a phase convention is a waste of time, as the phases cancel in summed-over quantities. All the ugly square roots, one remembers from quantum mechanics, come from sticking  into 3-j symbols. Wigner [158] 6-j *symbols* are related to our 6-j *coefficients* by


(5.15)

The name $3n-j$ coefficient comes from atomic physics, where a recoupling involves $3n$ angular momenta $j_1, j_2, \ldots, j_{3n}$.

Most of the textbook symmetries of and relations between 6-j symbols are obvious from looking at the corresponding diagrams; others follow quickly from completeness relations.

If we know the necessary 6-j's, we can compute the level splittings due to single particle exchanges. In the next section we shall show that a far stronger claim can be made: given the 6-j coefficients, we can compute *all* multiparticle matrix elements.

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	Skeletons	Vertex insertions	Self-energy insertions	Total number
1-j				1
3j				1
6-j				2
9-j				5
12-j				16

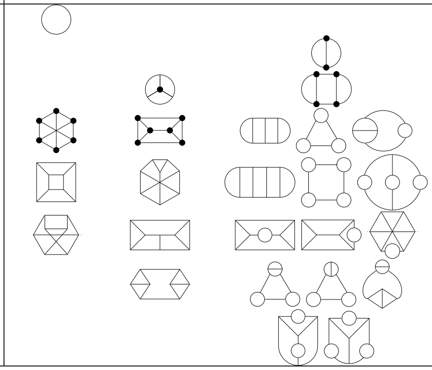
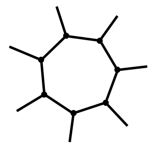


Table 5.1 Topologically distinct types of Wigner $3n$-j coefficients, enumerated by brute force (drawing all possible graphs, eliminated the topologically equivalent ones by hand). Lines meeting in any 3-vertex correspond to any 3 irreducible representations with a non-vanishing Clebsch-Gordan coefficient, so in general these graphs cannot be reduced to simpler graphs by means of such as the Lie algebra (4.45) and Jacobi identity (4.46).

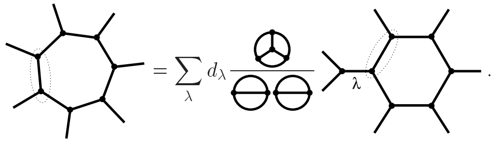
5.2 WIGNER $3n$-j COEFFICIENTS

An arbitrary higher order contribution to a 2-particle scattering process will give a complicated matrix element. The corresponding energy levels, cross-sections, *etc*, are expressed in terms of scalars obtained by contracting all tensor indices; diagrammatically they look like "vacuum bubbles", with $3n$ internal lines. The topologically distinct vacuum bubbles in low orders are given in table 5.1.

In group-theoretic literature, these diagrams are called $3n$-j symbols, and are studied in considerable detail. Fortunately, any $3n$-j symbol which contains as a sub-diagram a loop with, let us say, seven vertices

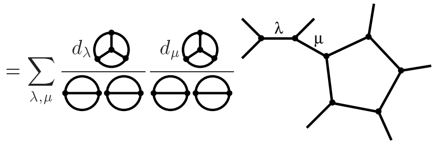


Replace the dotted pair of vertices by the cross-channel sum (5.13):



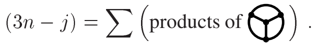
(5.16)

Now the loop has six vertices. Repeating the replacement for the next pair of vertices, we obtain a loop of length five:



(5.17)

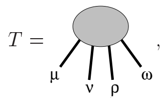
Repeating this process we can eliminate the loop altogether, producing 5-vertex-trees times bunches of 6-j coefficients. In this way we have expressed the original $3n$-j coefficients in terms of $3(n$-1)-j coefficients and 6-j coefficients. Repeating the process for the $3(n$-1)-j coefficients, we eventually arrive at the result that



(5.18)

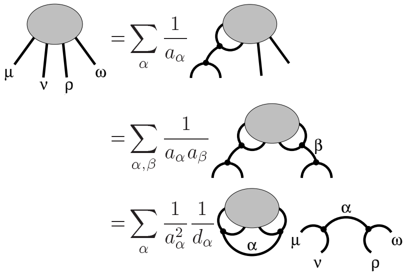
5.3 WIGNER-ECKART THEOREM

For concreteness, consider an arbitrary invariant tensor with four indices:



(5.19)

where μ, ν, ρ and ω are rep labels, and indices and line arrows are suppressed. Now insert repeatedly the completeness relation (5.8) to obtain



(5.20)

In the last line we have used the orthonormality of projection operators - as in (5.13) or (5.23).

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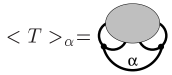
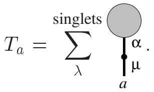
In this way any invariant tensor can be reduced to a sum over clebsches ("kinematics") weighted by "reduced matrix elements":



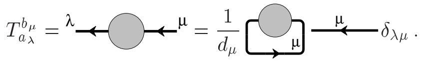
(5.21)

This theorem has many names, depending on how the indices are grouped. If T is a vector, then only the 1-dimensional reps (singlets) contribute



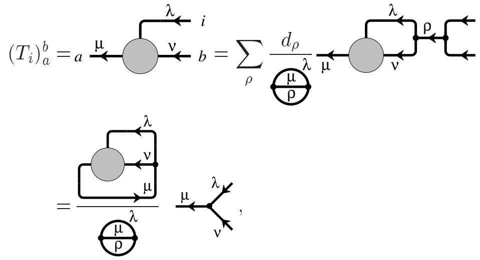
(5.22)

If T is a matrix, and the reps α, μ are irreducible, the theorem is called Schur's Lemma (for an irreducible rep an invariant matrix is either zero, or proportional to the unit matrix):



(5.23)

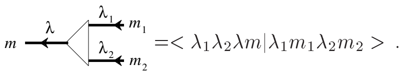
If T is an "invariant tensor operator", then the theorem is called the Wigner - Eckart theorem [158, 58]:



(5.24)

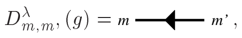
(assuming that μ appears only once in $\overline{\lambda} \otimes \mu$ Kronecker product). If T has many indices, as in our original example (5.19), the theorem is ascribed to Yutsis, Levinson and Vanagas [165]. The content of all these theorems is that they reduce spectroscopic calculations to evaluation of "vacuum bubbles" or "reduced matrix elements" (5.21).

The rectangular matrices $(C_\lambda)_\sigma^\alpha$ from (3.24) do not look very much like the clebsches from the quantum mechanics textbooks; neither does the Wigner-Eckart theorem in its birdtrack version (5.22). The difference is merely a difference of notation. In the bra-ket formalism, a clebsch for $\lambda_1 \otimes \lambda_2 \to \tilde{\lambda}$ is written as



(5.25)

Representing the $[d_\lambda \times d_\lambda]$ rep of a group element g diagrammatically by a black triangle



(5.26)

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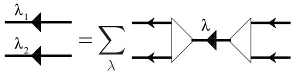
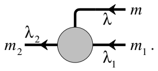
we can write the Clebsch-Gordan series (3.46) as



$$D^{\lambda_1}_{m_1 m_1'}(g) D^{\lambda_2}_{m_2 m_2'}(g) = \sum_{\tilde\lambda, \tilde m, \tilde m_1} < \lambda_1 m_1 \lambda_2 m_2 | \lambda_1 \lambda_2 \tilde\lambda \tilde m > D^{\tilde\lambda}_{\tilde m \tilde m_1}(g) < \lambda_1 \lambda_2 \tilde\lambda \tilde m_1 | \lambda_1 m_1' \lambda_2 m_2' > .$$

An "invariant tensor operator" can be written as

$$< \lambda_2 m_2 | T^\lambda_m | \lambda_1 m_1 > = \quad (5.27)$$



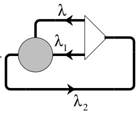
In the bra-ket formalism, the Wigner-Eckart theorem (5.24) is written as

$$< \lambda_2 m_2 | T^\lambda_m | \lambda_1 m_1 > = < \lambda \lambda_1 \lambda_2 m_2 | \lambda m \lambda_1 m_1 > T(\lambda, \lambda_1 \lambda_2) \,, \quad (5.28)$$

where the reduced matrix element is given by

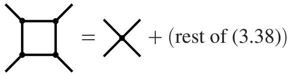
$$T(\lambda, \lambda_1 \lambda_2) = \frac{1}{d_{\lambda_2}} \sum_{n_1, n_2, n} < \lambda n \lambda_1 n_1 | \lambda \lambda_1 \lambda_2 n_2 > < \lambda_2 n_2 | T^\lambda_n | \lambda_1 n_1 >$$

$$= \frac{1}{d_{\lambda_2}} \quad (5.29)$$



We do not find the bra-ket formalism convenient for the group-theoretic calculations that will be discussed here.

There is a natural hierarchy to invariance groups, hinted at in sect. 3.5, that can perhaps already be grasped at this stage. Suppose we have constructed the invariance group G_1 which preserves primitives (3.36). Adding a new primitive, let us say a quartic invariant, means that we have imposed a new constraint; only those transformations of G_1 which also preserve the additional primitive constitute G_2, the invariance group of —, ⅄, ╳. Hence, G_2 is a subgroup of G_1, $G_2 \subseteq G_1$. Suppose now that you think that the primitiveness assumption is too strong, and that some quartic invariant, let us say (3.34), *cannot* be reduced to a sum of tree invariants (3.38), *ie.* it is of form



where ╳ is a new primitive, not included in the original list of primitives. By the above argument, $G_2 \subseteq G_1$. If G_1 does not exist (the invariant relations are so stringent that there is no space on which they can be realized).

Chapter Six

Permutations

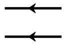
The simplest example of invariant tensors is the products of Kronecker deltas. On tensor spaces they represent index permutations. This is the way in which the symmetric group S_p, the group of permutations of p objects, enters into the theory of tensor reps. In this chapter, we introduce birdtracks notation for permutations, symmetrizations and antisymmetrizations and collect a few results which will be useful later on. These are the (anti)symmetrization expansion formulas (6.10) and (6.19), Levi-Civita tensor relations (6.28) and (6.31), the characteristic equations (6.51) and the invariance conditions (6.55) and (6.58).

6.1 SYMMETRIZATION

Operation of permuting tensor indices is a linear operation, and we can represent it by a $[d \times d]$ matrix:

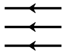
$$\sigma_\alpha^\beta = \sigma_{b_1 \ldots b_p}^{a_1 a_2 \ldots a_q}{}^{,d_p \ldots d_1}_{,c_q \ldots c_2 c_1} = \delta \qquad (6.1)$$

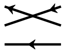
where $(\ldots)_\sigma$ stands for the desired permutation of indices. As the covariant and contravariant indices have to be permuted separately, it is sufficient to consider permutations of purely covariant tensors.
For 2-index tensors, there are two permutations

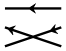
identity: $\mathbf{1}_{ab},{}^{cd} = \delta_a^d \delta_b^c =$ 

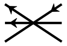
flip: $\sigma_{(12)ab},{}^{cd} = \delta_a^c \delta_b^d =$  (6.2)

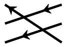
For 3-index tensors, there are six permutations

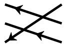
$\mathbf{1}_{a_1 a_2 a_3},{}^{b_3 b_2 b_1} = \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} =$ 

$\sigma_{(12)a_1 a_2 a_3},{}^{b_3 b_2 b_1} = \delta_{a_1}^{b_2} \delta_{a_2}^{b_1} \delta_{a_3}^{b_3} =$ 

$\sigma_{(23)} =$ 

$\sigma_{(13)} =$ 

$\sigma_{(123)} =$ 

$\sigma_{(132)} =$  (6.3)

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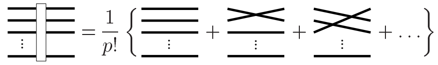
Subscripts refer to the standard cycle notation. (In the above, and for the remainder of this chapter, we shall usually omit the arrows on the Kronecker delta lines.)

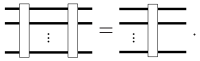
The symmetric sum of all permutations

$$S_{a_1 a_2 \ldots a_p},{}^{b_p \ldots b_2 b_1} = \frac{1}{p!}\left\{\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\ldots\delta_{a_p}^{b_p} + \delta_{a_2}^{b_1}\delta_{a_1}^{b_2}\ldots\delta_{a_p}^{b_p} + \ldots\right\}$$

$S =$  (6.4)

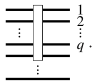
yields the symmetrization operator S. In birdtrack notation, a white bar drawn across p lines will always denote symmetrization of the lines crossed. Factor $1/p!$ has been introduced in order that S satisfies the projection operator normalization

$$S^2 = S$$

 (6.5)

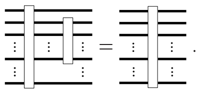
A subset of indices $a_1, a_2, \ldots a_q$, $q < p$ can be symmetrized by symmetrization matrix $S_{12\ldots q}$

$$(S_{12\ldots q})_{a_1 a_2 \ldots a_q \ldots a_p},{}^{b_p \ldots b_q \ldots b_2 b_1} = \frac{1}{q!}\left\{\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\ldots\delta_{a_q}^{b_q} + \delta_{a_2}^{b_1}\delta_{a_1}^{b_2}\ldots\delta_{a_q}^{b_q} + \ldots\right\}\delta_{a_q+1}^{b_q+1}\ldots\delta_{a_p}^{b_p}$$

$S_{12\ldots q} =$  (6.6)

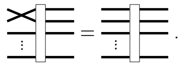
Overall symmetrization also symmetrizes any subset of indices:

$$S S_{12\ldots q} = S$$

 (6.7)

Any permutation has eigenvalue 1 on the symmetric tensor space:

$$\sigma S = S$$

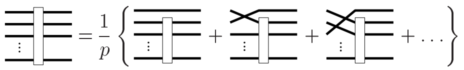
 (6.8)

Diagrammatically this means that legs can be crossed and un-crossed at will.

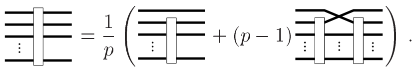
The definition (6.4) of the symmetrization operator as the sum of all $p!$ permutations is inconvenient for explicit calculations - a recursive definition is more useful:

$$S_{a_1 a_2 \ldots a_p},{}^{b_p \ldots b_2 b_1} = \frac{1}{p}\left\{\delta_{a_1}^{b_1} S_{a_2 \ldots a_p},{}^{b_p \ldots b_2} + \delta_{a_2}^{b_1} S_{a_1 a_3 \ldots a_p},{}^{b_p \ldots b_2} + \ldots\right\}$$

$$S = \frac{1}{p}\left(1 + \sigma_{(21)} + \sigma_{(321)} + \ldots + \sigma_{(p \ldots 321)}\right) S_{23 \ldots p}$$

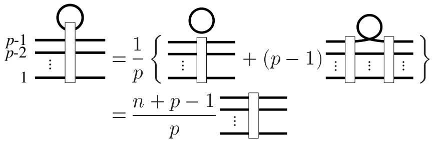
 (6.9)

which involves only p terms. This equation says, that if we start with the first index, we end up either with the first index, or the second index and so on. The remaining indices are fully symmetric. Multiplying by $S_{23}\ldots p$ from the left, we obtain an even more compact recursion relation with two terms only:



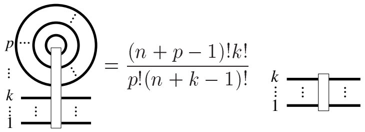
(6.10)

As a simple application, consider computation of a contraction of a single pair of indices:



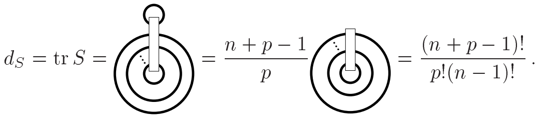
$$S_{a_p a_{p-1}\ldots a_1},{}^{b_1\ldots b_{p-1} a_p} = \frac{n+p-1}{p} S_{a_{p-1}\ldots a_1},{}^{b_1\ldots b_{p-1}} \,. \tag{6.11}$$

For a contraction in $(p-k)$ pairs of indices, we have



(6.12)

The trace of the symmetrization operator yields the number of independent components of fully symmetric tensors:



(6.13)

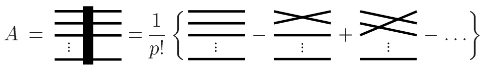
For example, for 2-index symmetric tensors

$$d_S = \frac{n(n+1)}{2} \,. \tag{6.14}$$

6.2 ANTISYMMETRIZATION

The alternating sum of all permutations

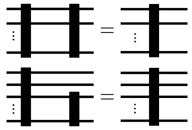
$$A_{a_1 a_2 \ldots a_p},{}^{b_p \ldots b_2 b_1} = \frac{1}{p!}\left\{\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\ldots\delta_{a_p}^{b_p} - \delta_{a_2}^{b_1}\delta_{a_1}^{b_2}\ldots\delta_{a_p}^{b_p} + \ldots\right\}$$



(6.15)

yields the antisymmetrization projection operator A. In birdtrack notation, antisymmetrization of p lines will always be denoted by a black bar drawn across the lines. As in the previous section

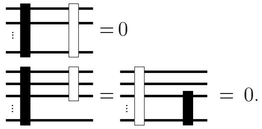
$$A^2 = A$$



(6.16)

and in addition

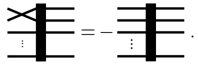
$$SA = 0$$



(6.17)

A transposition has eigenvalue -1 on the antisymmetric tensor space

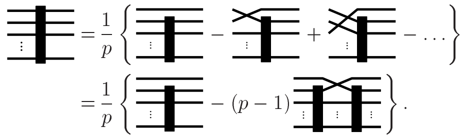
$$\sigma_{(i,i+1)} A = -A$$



(6.18)

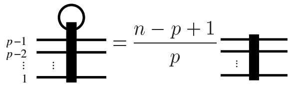
Diagrammatically this means that legs can be crossed and uncrossed at will, but with a factor of -1 for a transposition of any two neighboring legs.

As in the case of symmetrization operators, the recursive definition is often computationally convenient



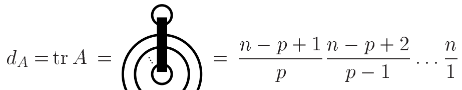
(6.19)

This is useful for computing contractions such as



$$A_{a a_{p-1}\ldots a_1},{}^{b_1\ldots b_{p-1} a} = \frac{n-p+1}{p} A_{a_{p-1}\ldots a_1},{}^{b_1\ldots b_{p-1}} \,. \tag{6.20}$$

The number of independent components of fully antisymmetric tensors is given by

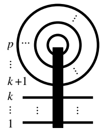


$$= \begin{cases} \frac{n!}{p!(n-p)!}\,, & n \geq p \\ 0\,, & n < p \end{cases} \,. \tag{6.21}$$

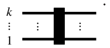
For example, for 2-index antisymmetric tensors the number of independent components is

$$d_A = \frac{n(n-1)}{2}\,. \tag{6.22}$$

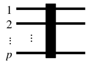
Tracing $(p-k)$ pairs of indices yields



$$= \frac{k!(n-k)!}{p!(n-p)!} \tag{6.23}$$

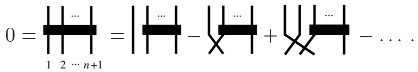


The antisymmetrization tensor $A_{a_1 a_2 \ldots ,}{}^{b_p \ldots b_2 b_1}$ has non-vanishing components, only if all lower (or upper) indices differ from each other. If the defining dimension is smaller than the number of indices, the tensor A has no non-vanishing components



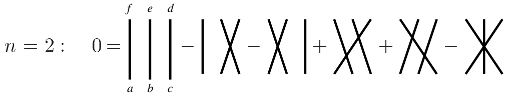
$$= 0 \qquad \text{if } p > n\,. \tag{6.24}$$

This identity implies that for $p > n$, not all combinations of p Kronecker deltas are linearly independent. A typical relation is the $p = n+1$ case



(6.25)

For example, for $n = 2$ we have



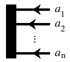
(6.26)

$$0 = \delta_a^f \delta_b^e \delta_c^d - \delta_a^f \delta_c^e \delta_b^d - \delta_b^f \delta_a^e \delta_c^d + \delta_b^f \delta_c^e \delta_a^d + \delta_c^f \delta_a^e \delta_b^d - \delta_c^f \delta_b^e \delta_a^d\,.$$

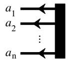
6.3 LEVI-CIVITA TENSOR

An antisymmetric tensor, with n indices in defining dimension n, has only one independent component ($d_n = 1$ by (6.21)). The clebsches (4.15) are in this case proportional to the Levi-Civita tensor

$$(C_A)_1{}^{,a_n \ldots a_2 a_1} = C\epsilon^{a_n \ldots a_2 a_1} =$$



$$(C_A)_{a_1 a_2 \ldots a_n}{}^{,1} = C\epsilon_{a_1 a_2 \ldots a_n} = \tag{6.27}$$



with $\epsilon^{12\ldots n} = \epsilon_{12\ldots n} = 1$. This diagrammatic notation for the Levi-Civita tensor was introduced by Penrose [132]. The normalization factors C are physically irrelevant. They adjust the phase and the overall normalization in order that the

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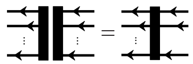
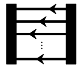
Levi-Civita tensors satisfy the projection operator (4.16) and orthonormality (4.17) conditions:

$$\frac{1}{N!}\epsilon_{b_1 b_2 \ldots b_n}\epsilon^{a_1 a_2 \ldots a_n} = A_{b_1 b_2 \ldots b_n,}{}^{a_n \ldots a_2 a_1}$$



(6.28)

$$\frac{1}{N!}\epsilon_{a_1 a_2 \ldots a_n}\epsilon^{a_1 a_2 \ldots a_n} = \delta_{11} = 1$$

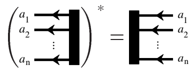


$$= 1\,. \tag{6.29}$$

With our conventions

$$C = \frac{i^{n(n-1)/2}}{\sqrt{n!}}. \tag{6.30}$$

The phase factor arises from the hermiticity condition (4.12) for clebsches (remember that indices are always read in the counterclockwise order around a diagram)



$$i^{-\phi}\epsilon_{a_1 a_2 \ldots a_n} = i^{-\phi}\epsilon_{a_n \ldots a_2 a_1}\,.$$

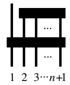
Transposing the indices

$$\epsilon_{a_1 a_2 \ldots a_n} = -\epsilon_{a_2 a_1 \ldots a_n} = \ldots = (-1)^{n(n-1)/2}\epsilon_{a_n \ldots a_2 a_1}\,,$$

yields $\phi = n(n-1)/2$. The factor $1/\sqrt{n!}$ is needed for the projection operator normalization (3.47).

Given n dimensions we cannot label more than n indices, so Levi-Civita tensors satisfy

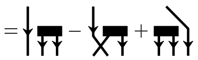
$$0 =$$



(6.31)

For example, for

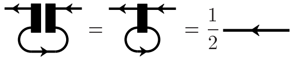
$$n = 2:\ 0 =$$



$$0 = \delta_a^d \epsilon_{bc} - \delta_b^d \epsilon_{ac} + \delta_c^d \epsilon_{ab}\,. \tag{6.32}$$

This is actually the same as the completeness relation (6.28), as can be seen by contracting (6.32) with ϵ_{cd} and using

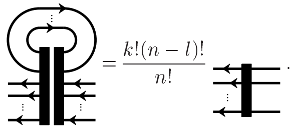
$$n = 2:$$



$$\epsilon_{ac}\epsilon^{bc} = \delta_a^b\,. \tag{6.33}$$

This relation is one of a series of relations obtained by contracting indices in the completeness relation (6.28) and substituting (6.23):

$$\epsilon_{a_n \ldots a_{k+1} b_k \ldots b_1} \epsilon^{a_n \ldots a_{k+1} a_k \ldots a_1} = (n-k)! k! \, A_{b_k \ldots b_1},^{a_1 \ldots a_k}$$



(6.34)

Such identities are familiar from relativistic calculations ($n = 4$):

$$\begin{aligned} \epsilon_{abcd}\epsilon^{agfe} &= \delta^{gfe}_{bcd} \\ \epsilon_{abcd}\epsilon^{abfe} &= 2\delta^{fe}_{cd} \\ \epsilon_{abcd}\epsilon^{abce} &= 6\delta^{e}_{d} \\ \epsilon_{abcd}\epsilon^{abcd} &= 24\,, \end{aligned} \tag{6.35}$$

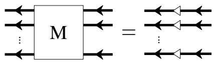
where the generalized Kronecker delta is defined by

$$\frac{1}{p!}\delta^{b_1 b_2 \ldots b_p}_{a_1 a_2 \ldots a_p} = A_{a_1 a_2 \ldots a_p},^{b_p \ldots b_2 b_1}\,. \tag{6.36}$$

6.4 DETERMINANTS

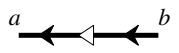
Consider an $[n^p \times n^p]$ matrix $M_\alpha{}^\beta$ defined by a direct product of $[n \times n]$ matrices M_a^b

$$M_\alpha{}^\beta = M_{a_1 a_2 \ldots a_p},^{b_p \ldots a_2 a_1} = M_{a_1}^{b_1} M_{a_2}^{b_2} \ldots M_{a_p}^{b_p}$$

$M =$ 

(6.37)

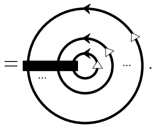
where

$M_a^b =$ 

(6.38)

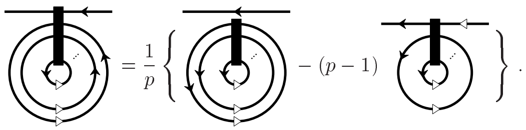
The trace of the antisymmetric projection of $M_\alpha{}^\beta$ is given by

$$\mathrm{tr}_{\,p} AM = A_{abc \ldots d},^{d' \ldots c' b' a'} M_{a'}^a M_{b'}^b \ldots M_{d'}^d$$



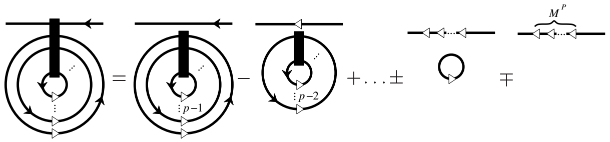
(6.39)

The subscript p on $\mathrm{tr}_{\,p}(\ldots)$ distinguishes the traces on $[n^p \times n^p]$ matrices M_α^β from the $[n \times n]$ matrix trace $\mathrm{tr}\, M$. To derive a recursive evaluation rule for $\mathrm{tr}_{\,p} AM$ use (6.19) to obtain



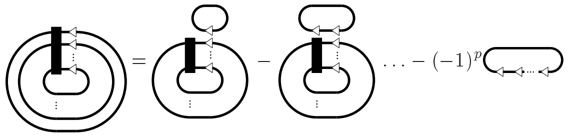
(6.40)

Iteration yields



(6.41)

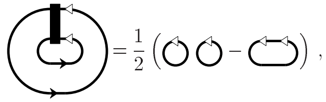
Contracting with M_a^b, we obtain



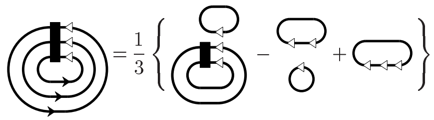
$$\mathrm{tr}_{\,p} AM = \frac{1}{p}\sum_{k=1}^{p}(-1)^{k-1}\left(\mathrm{tr}_{\,p-k} AM\right) \mathrm{tr}\, M^k\,. \tag{6.42}$$

This formula enables us to compute recursively all $\mathrm{tr}_{\,p} AM$ as polynomials in traces of powers of M:

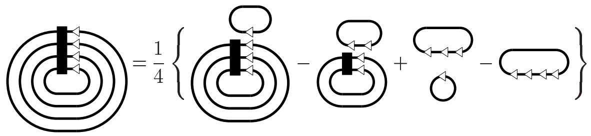
$\mathrm{tr}_{\,0} AM = 1\,,\qquad \mathrm{tr}_{\,1} AM =$  $= \mathrm{tr}\, M$ (6.43)



$$\mathrm{tr}_{\,2} AM = \frac{1}{2}\left\{(\mathrm{tr}\, M)^2 - \mathrm{tr}\, M^2\right\}\,, \tag{6.44}$$



$$\mathrm{tr}_{\,3} AM = \frac{1}{3!}\left\{(\mathrm{tr}\, M)^3 - 3(\mathrm{tr}\, M)(\mathrm{tr}\, M^2) + 2\,\mathrm{tr}\, M^3\right\}\,, \tag{6.45}$$



$$\begin{aligned} \mathrm{tr}_{\,4} AM = \frac{1}{4}\big\{&(\mathrm{tr}\, M)^4 - 6(\mathrm{tr}\, M)^2 \mathrm{tr}\, M^2 \\ &+ 3(\mathrm{tr}\, M^2)^2 + 8\,\mathrm{tr}\, M^3 \mathrm{tr}\, M - 6\,\mathrm{tr}\, M^4\big\}\,. \end{aligned} \tag{6.46}$$

For $p = n$ (M_a^b are $[n \times n]$ matrices) the antisymmetrized trace is the determinant

$$\det M = \mathrm{tr}_{\,n} AM = A_{a_1 a_2 \ldots a_n},{}^{b_n \ldots b_2 b_1} M_{b_1}^{a_1} M_{b_2}^{a_2} \ldots M_{b_n}^{a_n} . \qquad (6.47)$$

The coefficients in the above expansions are simple combinatoric numbers. A general term for $(\mathrm{tr}\, M^{\ell_1})^{\alpha_1} (\mathrm{tr}\, M^{\ell_2})^{\alpha_s}$, with α_1 loops of length ℓ_1, α_2 loops of length ℓ_2 and so on, is divided by the number of ways in which this pattern may be obtained:

$$\ell_1^{\alpha_1} \ell_2^{\alpha_2} \ldots \ell_s^{\alpha_s} \alpha_1! \alpha_2! \ldots \alpha_s! \, . \qquad (6.48)$$

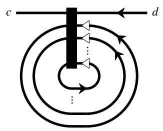
6.5 CHARACTERISTIC EQUATIONS

We have noted that the dimension of the antisymmetric tensor space is zero in $n < p$. This is rather obvious; antisymmetrization allows each label to be used at most once, and it is impossible to label more legs than there are labels. In terms of the antisymmetrization operator this is given by the identity

$$A = 0 \qquad \text{if } \; p > n \, . \qquad (6.49)$$

This trivial identity has an important consequence: it guarantees that any $[n \times n]$ matrix satisfies a characteristic (or Hamilton-Cayley) equation. Take $p = n+1$ and contract with M_a^b n index pairs of A:

$$A_{c a_1 a_2 \ldots a_n},{}^{b_n \ldots b_2 b_1 d} M_{b_1}^{a_1} M_{b_2}^{a_2} \ldots M_{b_n}^{a_n} = 0$$



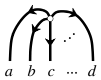
$= 0 \, . \qquad (6.50)$

We have already expanded this in (6.41). For $p = n+1$ we obtain the *characteristic equation*

$$0 = \sum_{k=0}^{n} (-1)^k (\mathrm{tr}_{\,n-k} AM) M^k \, , \qquad (6.51)$$
$$= M^n - (\mathrm{tr}\, M) M^{n-1} + (\mathrm{tr}_{\,2} AM)\, M^{n-2} - \ldots + (-1)^n (\det M)\, \mathbf{1} \, .$$

6.6 FULLY (ANTI)SYMMETRIC TENSORS

As we shall often use fully symmetric and antisymmetric tensors, it is convenient to introduce special birdtrack symbols for them. We shall denote a fully *symmetric* tensor by a small circle (white dot)

$d_{abc \ldots f} =$  . (6.52)

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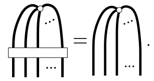
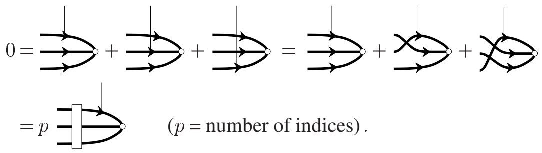
A symmetric tensor $d_{abc \ldots d} = d_{bac \ldots d} = d_{acb \ldots d} = \ldots$ satisfies

$$Sd = d$$



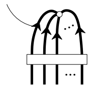
(6.53)

If this tensor is also an invariant tensor, the invariance condition (4.35) can be written as

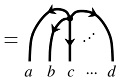


(p = number of indices) . (6.54)

Hence, the invariance condition for symmetric tensors is

$0 =$  . (6.55)

The fully *antisymmetric* tensors with *odd* numbers of legs will be denoted by black dots

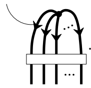
$f_{abc \ldots d} =$  . (6.56)

If the number of legs is *even*, an antisymmetric tensor is *anticyclic*

$$f_{abc \ldots d} = -f_{bc \ldots da} \, , \qquad (6.57)$$

and the birdtrack notation must distinguish the first leg. A black dot is inadequate for the purpose. A bar, as for the Levi-Civita tensor (6.27), a semicircle (the symplectic wart introduced below in (12.3)) or a similar notation fixes the problem.

For antisymmetric tensors, the invariance condition can be stated compactly as

$0 =$  . (6.58)

Chapter Seven

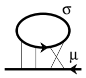
Casimir operators

The construction of invariance groups, developed elsewhere in this monograph, is self-contained, and none of the material covered in this chapter is necessary for understanding the remainder of the monograph. We have argued in sect. 5.2 that all relevant group-theoretic numbers are given by vacuum bubbles (reduced matrix elements, $3n$-j coefficients, *etc*), and we have described the algorithms for their evaluation. That is all that is really needed in applications.

However, one often wants to cross-check one's calculation against the existing literature. In this chapter we discuss why and how one introduces casimirs (or Dynkin indices), we construct independent Casimir operators for the classical groups, and finally we compile values of a few frequently used casimirs.

Our approach emphasizes the role of primitive invariants in constructing reps of Lie groups. Given a list of primitives, we present a systematic algorithm for constructing invariant matrices M_i and the associated projection operators (3.45).

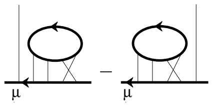
In the canonical, Cartan-Killing approach one faces a somewhat different problem. Instead of the primitives, one is given the generators T_i explicitly and no other invariants. Hence, the invariant matrices M_i can be constructed only from contractions of generators; typical examples are matrices

$M_2 =$  , $M_4 =$  , $\ldots$ (7.1)

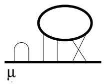
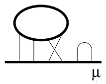
where σ, μ could be any reps, reducible or irreducible. Such invariant matrices are called *Casimir operators*.

What is a minimal set of Casimir operators, sufficient to reduce any rep to its irreducible subspaces? (Such bases can be useful, as the corresponding r Casimir operators uniquely label each irreducible rep by their eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_r$).

The invariance condition for any invariant matrix (3.28) is

$0 = [T_i, M] =$ 

so all Casimir operators commute

$M_2 M_4 =$  $=$  $= M_4 M_2\,,$

and, according to sect. 3.5, can be used to simultaneously decompose the rep μ. If $M_1, M_2 \ldots$ have been used in the construction of projection operators (3.45),

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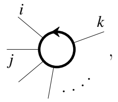
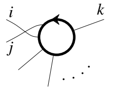
any matrix polynomial $f(M_1, M_2 \ldots)$ takes value $f(\lambda_1, \lambda_2, \ldots)$ on the irreducible subspace projected by P_i, so polynomials in M_i induce no further decompositions. Hence, it is sufficient to determine the finite number of M_i's which form a polynomial basis for all Casimir operators (7.1). Furthermore, as we show in the next section, it is sufficient to restrict the consideration to the symmetrized casimirs. This observation enables us to explicitly construct, in sect. 7.2, a set of independent casimirs for each classical group. Exceptional groups pose a more difficult challenge.

7.1 CASIMIRS AND LIE ALGEBRA

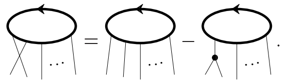
There is no general agreement on a unique definition of a Casimir operator. We could choose to call the trace of a product of k generators

$\mathrm{tr}\,(T_i T_j \ldots T_k) =$  , (7.2)

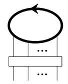
a kth order *casimir*. With such definition

$\mathrm{tr}\,(T_j T_i \ldots T_k) =$ 

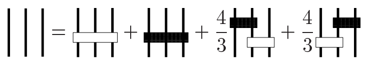
would also be a casimir, independent of the first one. However, all traces of T_i's which differ by a permutation of indices are related by Lie algebra. For example

 . (7.3)

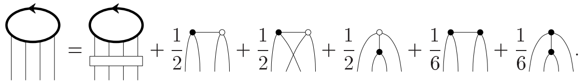
The last term involves a (k-1)th order casimir and is antisymmetric in the i, j indices. Only the fully symmetrized traces

$h_{ij\ldots k} \equiv \frac{1}{p!}\sum_{perm} \mathrm{tr}\,(T_i T_j \ldots T_k) =$  (7.4)

are not affected by the Lie algebra relations. Hence from now on, we shall use the term "casimir" to denote *symmetrized* traces (ref. [116] follows the same usage, for example). Any unsymmetrized trace $\mathrm{tr}\,(T_i T_j \ldots T_k)$ can be expressed in terms of the symmetrized traces. For example, using the symmetric group identity (see table 9.1)

 , (7.5)

the Jacobi identity (4.46), and the d_{ijk} definition (9.79), we can express the trace of four generators in any rep of any semi-simple Lie group in terms of the quartic and cubic casimirs:

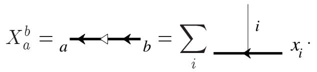


(7.6)

In this way, an arbitrary kth order trace can be written as a sum over tree contractions of casimirs. The symmetrized casimirs (7.4) are conveniently manipulated as monomial coefficients:

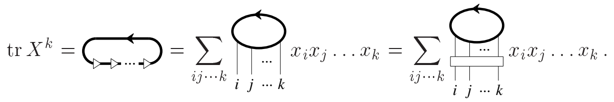
$$\mathrm{tr}\, X^k = h_{ij\ldots m} x_i x_j \ldots x_m . \qquad (7.7)$$

For a rep λ, X is a $[d_\lambda \times d_\lambda]$ matrix $X = x_i T_i$, where x_i is an arbitrary N-dimensional vector. We shall also use a birdtrack notation (6.38):



(7.8)

The symmetrization (7.4) is automatic



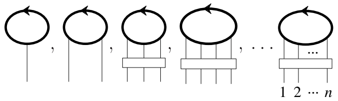
(7.9)

7.2 INDEPENDENT CASIMIRS

Not all $\mathrm{tr}\, X^k$ are independent. For n-dimensional rep a typical relation relating various $\mathrm{tr}\, X^k$ is the characteristic equation (6.51):

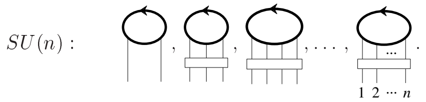
$$X^n = (\mathrm{tr}\, X) X^{n-1} - (\mathrm{tr}_2 X) X^{n-2} + \ldots \pm (\det X) . \qquad (7.10)$$

Scalar coefficients $\mathrm{tr}_k X$ are polynomials in $\mathrm{tr}\, X^m$, computed in sect. 6.5. The characteristic equation enables us to express any $X^p, p \geq n$ in terms of the matrix powers $X^k, k < n$ and the scalar coefficients $\mathrm{tr}\, X^k, k \leq n$. If a group has an $[n \times n]$ dimensional rep, it has at most n independent casimirs



corresponding to $\mathrm{tr}\, X, \mathrm{tr}\, X^2, \mathrm{tr}\, X^3, \ldots \mathrm{tr}\, X^n$.

For a simple Lie group, the number of independent casimirs is called the *rank* of the group and is always smaller than n, the dimension of the lowest dimensional rep. For example, for all simple groups $\mathrm{tr}\, T_i = 0$, the first casimir is always identically zero. For this reason, the rank of $SU(n)$ is $n-1$, and the independent casimirs are



(7.11)

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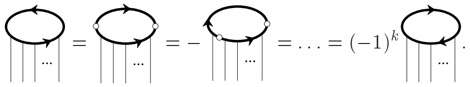
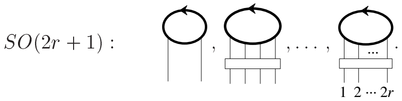
The defining reps of $SO(n)$, $Sp(n)$, G_2, F_4, E_7 and E_8 groups have invertible bilinear invariant g_{ab}, either symmetric or skew-symmetric. Inserting $\delta_a^c = g_{ab} g^{bc}$ any place in a trace of k generators, and moving the tensor g_{ab} through the generators by means of the invariance condition (10.5), we can reverse the defining rep arrow:



(7.12)

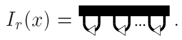
Hence for the above groups, $\mathrm{tr}\, X^k = 0$ for k odd, and all their casimirs are of even order.

The odd and the even dimensional orthogonal groups differ in the orders of independent casimirs. For $n = 2r + 1$, there are r independent casimirs



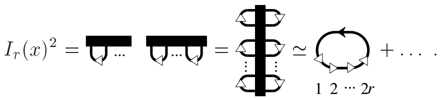
(7.13)

For $n = 2r$, a symmetric invariant can be formed by contracting r defining reps with a Levi-Civita tensor (the adjoint projection operator (10.13) is antisymmetric):



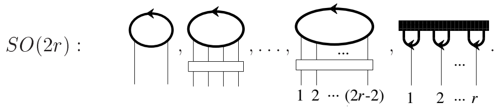
(7.14)

$\mathrm{tr}\, X^{2r}$ is not independent, as by (6.28), it is contained in the expansion of $I_r(x)^2$



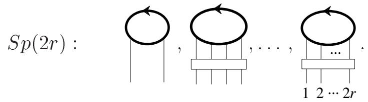
(7.15)

Hence, the r independent casimirs for even dimensional orthogonal groups are:



(7.16)

For $Sp(2r)$ there are no special relations, and the r independent casimirs are $\mathrm{tr}\, X^{2k}, 0 < l \leq r$;



(7.17)

The characteristic equation (7.10), by means of which we count the independent casimirs, applies to the lowest dimensional rep of the group, and one might worry that other reps might be characterized by further independent casimirs. The answer is no; all casimirs can be expressed in terms of the defining rep. For $SU(n)$, $Sp(n)$ and $SO(n)$ tensor reps this is obvious from the explicit form of the generators in higher reps (see sect. 9.4 and related results for $Sp(n)$ and $SO(n)$); they are

A_r	2, 3, …, $r+1$	$\sim$	$SU(r+1)$
B_r	2, 4, 6, …, $2r$	$\sim$	$SO(2r+1)$
C_r	2, 4, 6, …, $2r$	$\sim$	$Sp(2r)$
D_r	2, 4, …, $2r-2$, r	$\sim$	$SO(2r)$
G_2	2, 6		
F_4	2, 6, 8, 12		
E_6	2, 5, 6, 8, 9, 12		
E_7	2, 6, 8, 10, 12, 14, 18		
E_8	2, 8, 12, 14, 18, 20, 24, 30		

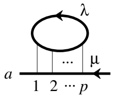
Table 7.1 Betti numbers for the simple Lie groups.

tensor products of the defining rep generators and Kronecker deltas, and a higher rep casimir always reduces to sums of the defining rep casimirs, times polynomials in n (see examples of sect. 9.6).

For the exceptional groups, cubic and higher defining rep invariants enter, and the situation is not so trivial. We shall show below, by explicit computation, that $\mathrm{tr}\, X^3 = 0$ for E_6 and $\mathrm{tr}\, X^4 = c(\mathrm{tr}\, X^2)^2$ for all exceptional groups. We shall also prove the reduction to the 2^{nd}- and 6^{th}-order casimirs for G_2 and partially prove the reduction for other exceptional groups. The orders of all independent casimirs are known [26] as the Betti numbers, listed here in table 7.1.

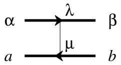
7.3 CASIMIR OPERATORS

Most physicists would not refer to $\mathrm{tr}\, X^k$ as a casimir. Casimir's [23] quadratic operator and its generalizations [136] are $[d_\mu \times d_\mu]$ matrices

$(I_p)_a^b =$


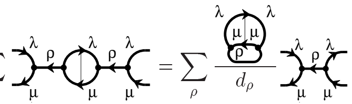
$= [\mathrm{tr}_\lambda(T_k \dots T_i T_j)]\,(T_i T_j \dots T_k)_a^b . \qquad (7.18)$

We have shown in sect. 5.2 that all invariants are reducible to $6j$ coefficients. I_p's are particularly easy to express in terms of $6j$'s. Define

$M_{b}^{\alpha}{}_{,}{}^{\mu}_{\beta} =$


$\alpha, \beta = 1, \dots, d_\lambda , \quad a, b = 1, 2, \dots, d_\mu . \qquad (7.19)$

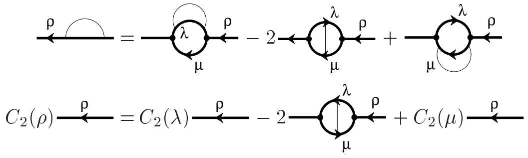
Inserting the complete Clebsch-Gordan series (5.8) for $\lambda \otimes \mu$, we obtain

$M = \sum_\rho$


$. \qquad (7.20)$

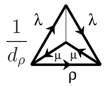
The eigenvalues of M are Wigner's $6j$ coefficients (5.15). It is customary to express these $6j$'s in terms of quadratic casimir operators by using the invariance condition

(4.39)



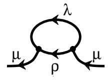
$. \qquad (7.21)$

This is an ancient formula familiar from quantum mechanics textbooks: if the total angular momentum is $J = L + S$, then the cross-term $L \cdot S = \frac{1}{2}(J^2 - L^2 - S^2)$. In the present case we trace both sides to obtain



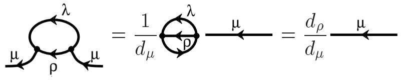
$= -\frac{1}{2}\left\{C_2(\rho) - C_2(\lambda) - C_2(\mu)\right\} . \qquad (7.22)$

The pth order casimir is thus [117]

$$
\begin{aligned}
(I_p)_a^b &= (M^p)_{a\alpha}^{\alpha b} \\
&= \sum_\rho^{irreduc.} \left(\frac{C_2(\rho) - C_2(\lambda) - C_2(\mu)}{2}\right)^p
\end{aligned}
$$


.

If μ is an irreducible rep, (5.23) yields



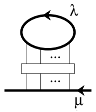
,

and the μ rep eigenvalue of I_p is given by

$$\sum_\rho \left(\frac{C_2(\rho) - C_2(\lambda) - C_2(\mu)}{2}\right)^p d_\rho . \qquad (7.23)$$

Here the sum goes over all $\rho \subset \lambda \otimes \mu$, where ρ, λ and μ are irreducible reps.

Another definition of the generalized Casimir operator, which is more in the spirit of the previous section, uses the fully symmetrized trace:



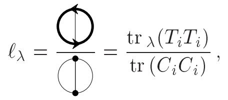
$= h(\lambda)_{ij\dots k}(T_i T_j \dots T_k)_a^b . \qquad (7.24)$

We shall return to this definition in the next section.

7.4 DYNKIN INDICES

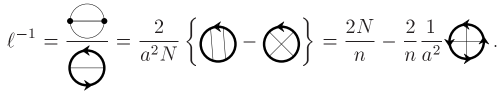
As we have seen so far, there are many ways of defining casimirs; in practice it is usually quicker to directly evaluate a given birdtrack diagram than to relate it to standard casimirs. Still, it is good to have an established convention, if for no other reason than to be able to cross-check one's calculation against the tabulations available in the literature.

Usually a rep is specified by its dimension. If the group has several inequivalent reps with the same dimensions, further numbers are needed to uniquely determine the rep. Specifying the *Dynkin index* [56],



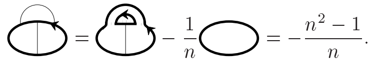
(7.25)

usually (but not always) does the job. A Dynkin index is easy to evaluate by birdtrack methods. By the Lie algebra (4.45), the defining rep Dynkin index is related to a $6j$ coefficient:



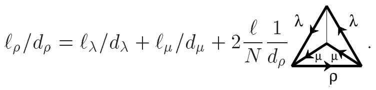
(7.26)

The $6j$ coefficient  $= \mathrm{tr}\,(T_i T_j T_i T_j)$ is evaluated by the usual birdtrack tricks. For $SU(n)$, for example



(7.27)

The Dynkin index of a rep ρ in the Clebsch-Gordan series for $\lambda \otimes \mu$ is related to a $6j$ coefficient by (7.22):



(7.28)

We shall usually evaluate Dynkin indices by this relation. Another convenient formula for evaluation of Dynkin indices for semi-simple groups is

$$\ell_\lambda = \frac{\mathrm{tr}_{\,\lambda} X^2}{\mathrm{tr}_{\,A} X^2} \,. \tag{7.29}$$

An application of this formula is given in sect. 9.6.

The form of the Dynkin index is motivated by a few simple considerations. First, we want an invariant number, so we trace all indices. Second, we want a pure, normalization independent number, so we take a ratio. $\mathrm{tr}\,(C_i C_i)$ is the natural normalization scale, as all groups have the adjoint rep. Furthermore, unlike the Casimir operators (7.18) which have single eigenvalues $I_p(\lambda)$ only for irreducible reps, the Dynkin index is a pure number for both reducible and irreducible reps. [Exercise: compute the Dynkin index for $U(n)$.]

The above criteria lead to the Dynkin index as the unique group-theoretic scalar corresponding to the quadratic Casimir operator. The choice of group- theoretic scalars corresponding to higher casimirs is rather more arbitrary. Consider the

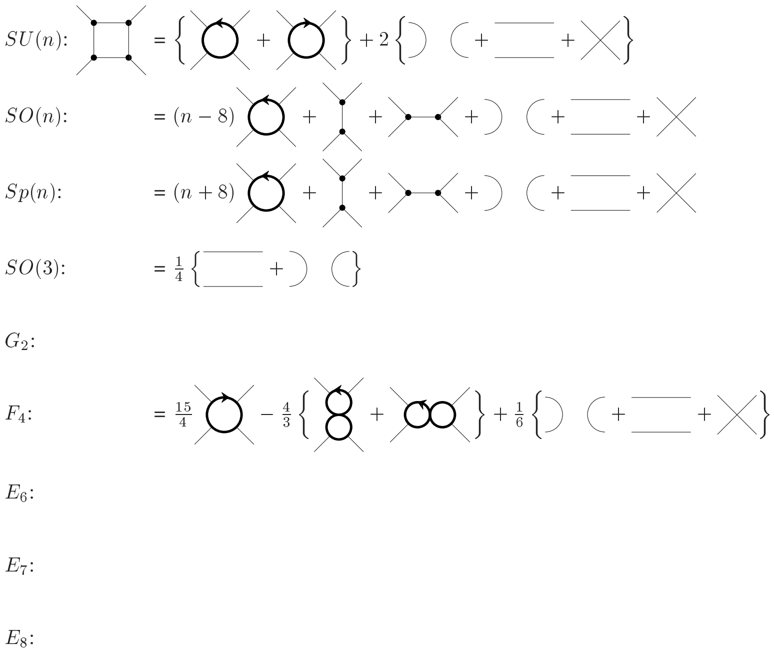


Table 7.2 Expansions of the adjoint rep quartic casimirs in terms of the defining rep for all simple Lie algebras. The normalization (7.37) is set to $a = 1$.

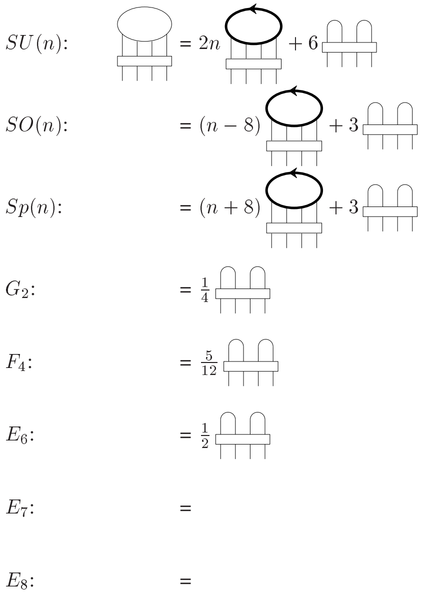
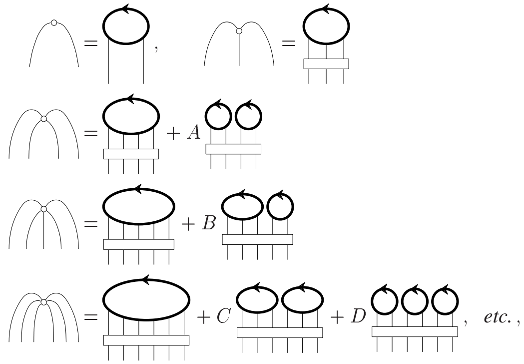


Table 7.3 Reduction of adjoint quartic casimirs to the defining rep quartic casimirs for all simple Lie algebras. The normalization (7.37) is set to $a = 1$.

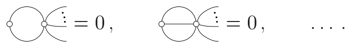
reductions of I_4 for the adjoint reps, tabulated in table 7.2. (The $SU(n)$ was evaluated as an introductory example, sect. 2.2; the remaining examples are evaluated by inserting the appropriate adjoint projection operators, derived below).

Quartic casimirs contain quadratic bits, and in general, expansions of $h(\lambda)$'s in terms of the defining rep will contain lower order casimirs. To construct the "pure" pth order casimirs, we introduce



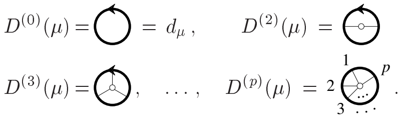
(7.30)

and fix the constants $A, B, C, \ldots$ by requiring that these casimirs are *orthogonal*:



(7.31)

Now we can define the *generalized Dynkin indices* [121] by

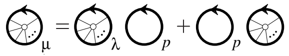


(7.32)

For simplicity, we have taken here normalization $\mathrm{tr}\,(C_i C_i) = 1$.

The generalized Dynkin indices are not particularly convenient or natural from the computational point of view, but they do have some nice properties. For example (as we shall show later on), the exceptional groups $\mathrm{tr}\, X^4 = C(\mathrm{tr}\, X^2)^2$ are singled our by $D^{(4)} = 0$.

If μ is a Kronecker product of two reps, $\mu = \lambda \otimes \rho$, the generalized Dynkin indices satisfy



$$D^{(p)}(\mu) = D^{(p)}(\lambda) d_\rho + d_\lambda D^{(p)}(\rho) > 0 \,, \qquad (7.33)$$

as the cross terms vanish by the orthonormality conditions (7.31). Substituting the completeness relation (5.7), $\lambda \otimes \rho = \sum \sigma$, we obtain a family of *sum rules* for the generalized Dynkin indices:

$$\sum_\sigma$$  $$= \sum_\sigma D^{(p)}(\sigma) = D^{(p)}(\lambda) d_\rho + d_\lambda D^{(p)}(\rho). \qquad (7.34)$$

For $p = 2$ this is a $\lambda \otimes \rho = \sum \sigma$ sum rule for Dynkin indices (7.27)

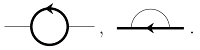
$$\ell_\lambda d_\rho + d_\lambda \ell_\rho = \sum_\sigma \ell_\sigma \,, \tag{7.35}$$

useful in checking the correctness of Clebsch-Gordan decompositions.

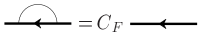
7.5 QUADRATIC, CUBIC CASIMIRS

As the low-order Casimir operators appear so often in physics, it is useful to list them and their relations.

Given two generators T_i, T_j in $[n\times n]$ rep λ, there are only two ways to form a loop:

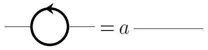


If the λ rep is irreducible, we define C_F casimir as



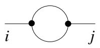
$$(T_iT_i)_a^b = C_F\delta_a^b. \tag{7.36}$$

If the adjoint rep is irreducible, we define



$$\mathrm{tr}\, T_iT_j = a\,\delta_{ij}. \tag{7.37}$$

Usually we take λ to be the defining rep and fix the overall normalization by taking $a = 1$. For the adjoint rep (dimension N), we use notation

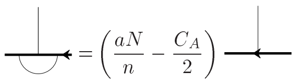


$$= C_{ik\ell}C_{jk\ell} = C_A \,. \tag{7.38}$$

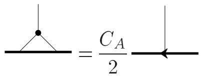
C_F, a, C_A, and ℓ, the Dynkin index (7.27), are related by tracing the above expressions:

$$= nC_F = Na = NC_A\ell. \tag{7.39}$$

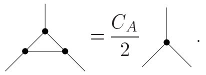
While the Dynkin index is normalization independent, one of C_F, a or C_A has to be fixed by a convention. The cubic casimirs formed from T_i's and C_{ijk}'s are (all but one) reducible to the quadratic Casimir operators:



(7.40)



(7.41)



(7.42)

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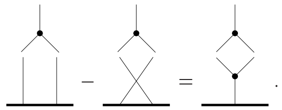
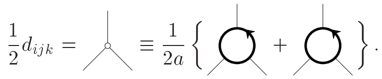
This follows from the Lie algebra (4.45)



The one exception is the symmetrized third-order casimir

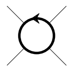


(7.43)

By (7.12) this vanishes for all groups whose defining rep is not complex. That leaves behind only $SU(n), n \geq 3$ and E_6. As we shall show, in sect. 18.6, $d_{ijk} = 0$ for E_6, only $SU(n)$ groups have non-vanishing cubic casimirs.

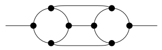
7.6 QUARTIC CASIMIRS

There is no unique definition of a quartic casimir. Any group-theoretic weight which contains a trace of four generators



(7.44)

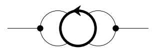
can be called a quartic casimir. For example, 4-loop contribution to the QCD β function



(7.45)

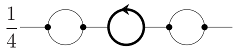
contains two quartic casimirs. This weight cannot be expressed as a function of quadratic casimirs and has to be computed separately for each rep and each group. For example, such quartic casimirs need to be evaluated for the purpose of classification of grand unified theories [117], weak coupling expansions in lattice gauge theories [43], and the classification of reps of simple Lie algebras [109].

However, not every birdtrack diagram, which contains a trace of four generators, is a genuine quartic casimir. For example,



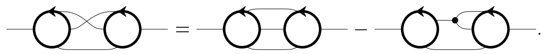
(7.46)

is reducible by (7.41) to



(7.47)

and equals $\frac{1}{4}aC_A^2$ for a simple Lie algebra. However, if all loops contain four vertices or more, Lie algebra cannot be used to reduce the diagram. For example



(7.48)

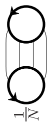
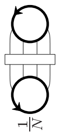
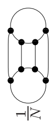
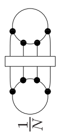
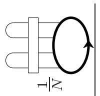
					
$U(n)$	n^2	$\frac{n^2+5}{6}$	$2n^2(n^2+12)$	$\frac{2n^2(n^2+36)}{3}$	$\frac{2n^2+1}{3n}$
$SU(n)$	$\frac{n^4-3n^2+3}{n^2}$	$\frac{n^4-6n^2+18}{6n^2}$	$2n^2(n^2+12)$	$\frac{2n^2(n^2+36)}{3}$	$\frac{2n^2-3}{3n}$
$SO(n)$	$\frac{n^2-3n+4}{8}$	$\frac{n^2-n+4}{24}$	$\frac{(n-2)(n^3-9n^2+54n-104)}{8}$	$\frac{(n-2)(n^3-15n^2+138n-296)}{24}$	$\frac{2n-1}{6}$
$Sp(n)$	$\frac{n^2+3n+4}{8}$	$\frac{n^2+n+4}{24}$	$\frac{(n+2)(n^3+9n^2+54n+104)}{8}$	$\frac{(n+2)(n^3+15n^2+138n+296)}{24}$	$\frac{2n+1}{6}$
$G_2(7)$	$\frac{5}{3}$	$\frac{1}{3}$	$\frac{164}{3}$	$\frac{100}{3}$	$\frac{4}{3}$
$F_4(26)$	$\frac{7}{8}$	$\frac{1}{8}$	$\frac{79}{8}$	$\frac{25}{8}$	$\frac{3}{2}$
$E_6(27)$	$\frac{41}{27}$	$\frac{5}{27}$	28	$\frac{20}{3}$	$\frac{20}{9}$
$E_7(56)$	$\frac{53}{64}$	$\frac{5}{64}$	8	$\frac{320}{81}$	$\frac{15}{8}$
$E_8(248)$	$\frac{11}{120}$	$\frac{1}{120}$	$\frac{11}{120}$	$\frac{1}{120}$	$\frac{5}{6}$

Table 7.4 Various quartic casimirs for all simple Lie algebras. The normalization in (7.37) is set to $a = 1$.

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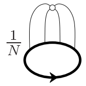
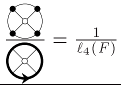
		 $= \frac{1}{\ell_4(F)}$
$SU(n)$	$\frac{(n^2-4)(n^2-9)}{6(n^2+1)}$	$2n$
$SO(n)$	$\frac{(n+1)(n^2-4)(n-3)}{24(n^2-n+4)}$	$n-8$
$Sp(n)$	$\frac{(n-1)(n^2-4)(n+3)}{24(n^2+n+4)}$	$n+8$

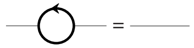
Normalization: 

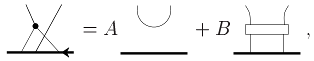
Table 7.5 Quartic Dynkin indices (7.32) for the defining and the adjoint reps of classical groups. For the exceptional groups the quartic Dynkin indices vanish identically.

The second diagram is reducible, but the first one is not. Hence, at least one quartic casimir from a family of quartic casimirs related by Lie algebra has to be evaluated directly. For the classical groups this is a straightforward application of the birdtrack reduction algorithms. For example, for $SU(n)$ we worked this out in sect. 2.2.

The results listed in table 7.4 for the defining and adjoint reps of all simple Lie groups. In table 7.5 we have used the results of table 7.4 to compute the quartic Dynkin indices (7.32). These computations were carried out by the methods which will be developed in the remainder of this monograph.

7.7 SUNDRY RELATIONS BETWEEN QUARTIC CASIMIRS

In evaluations of group theory weights the following reduction of a 2-adjoint, 2-defining indices quartic casimir is often very convenient:

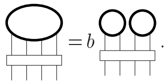


(7.49)

where the constants A and B are listed in table 7.5.

For the exceptional groups, the calculation of quartic casimirs is very simple. As mentioned above, the exceptional groups have no genuine quartic casimirs, as

$$\mathrm{tr}\, X^4 = b(\mathrm{tr}\, X^2)^2$$



(7.50)

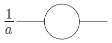
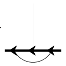
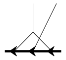
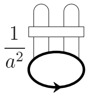
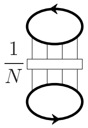
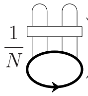
	N	$\frac{1}{a}$ 	$\frac{1}{a}$ 	
$SU(n)$	n^2-1	$2n$	$-\frac{1}{n}$	$-a^2$
$SO(n)$	$\frac{n(n-1)}{2}$	$(n-2)$	$\frac{1}{2}$	$-\frac{a^2}{2}$ $+a$
$Sp(n)$	$\frac{n(n+1)}{2}$	$(n+2)$	$\frac{1}{2}$	$-\frac{a^2}{2}$ $-a$
$G_2(7)$	14	4	0	$-\frac{a^2}{3}$ $+\frac{a}{3}$
$F_4(26)$	52	3	$\frac{1}{2}$	$-\frac{a^2}{12}$ $+\frac{a}{3}$
$E_6(27)$	78	4	$\frac{8}{9}$	$-\frac{a^2}{9}$ $+a$
$E_7(56)$	133	3	$\frac{7}{8}$	$-\frac{a^2}{24}$ $+\frac{5a}{6}$

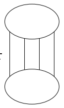
Table 7.6 The dimension N of the adjoint rep, the quadratic casimir of the adjoint rep $1/\ell$, the vertex casimir, C_v, and the quartic casimir (7.49) for all simple Lie algebras.

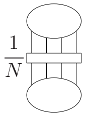
The constant is fixed by contracting with :

$$b = \frac{3}{N(N+2)}\frac{1}{a^2}$$  $$= \frac{3}{N(N+2)}\left(\frac{N}{n} - \frac{1}{6}\frac{C_A}{a}\right).$$

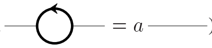
Hence, for the exceptional groups

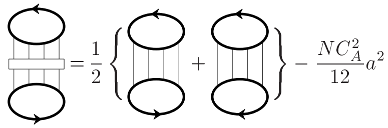
$\frac{1}{N}$  $= \frac{3}{N+2}\Bigg(\frac{1}{N}$  $\Bigg)^2 = \frac{3a^4}{N+2}\left(\frac{N}{n} - \frac{C_A}{6a}\right)\,,$ (7.51)

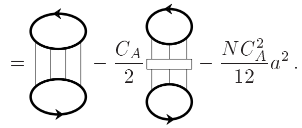
$\frac{1}{N}$  $= C_A^4 \frac{25}{12(N+2)}\,,$ (7.52)

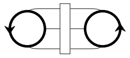
$\frac{1}{N}$  $= C_A^4 \frac{N+27}{12(N+2)}\,.$ (7.53)

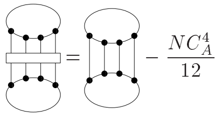
Here the third relation follows from the second by the Lie algebra.

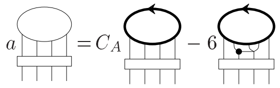
To facilitate such computations, we list a selection of relations between various quartic casimirs (using normalization ) for irreducible reps

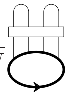
 (7.54)

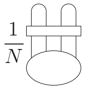
 (7.55)

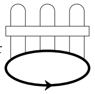
The cubic casimir  is non-vanishing only for $SU(n), n \geq 3$.

 (7.56)

 (7.57)

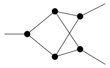
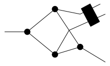
$$\frac{1}{a^2 N}$$  $$= \frac{1}{3a}(2C_F + C_V) = \frac{N}{n} - \frac{C_A}{6a} \tag{7.58}$$

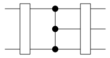
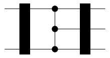
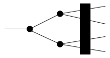
$$\frac{1}{N}$$  $$= \frac{5}{6} C_A^2 \tag{7.59}$$

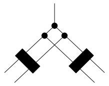
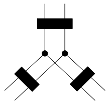
$$\frac{1}{a^3 N}$$  $$= \frac{1}{3}(C_F^2 + C_F C_V + C_V^2)\,. \tag{7.60}$$

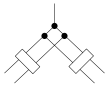
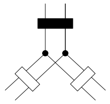
7.8 IDENTICALLY VANISHING TENSORS

There exists an interesting class of group theoretic weights which vanish *identically*. Some examples are

 $\equiv 0\,,$  $\equiv 0\,,$ (7.61)

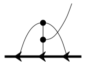
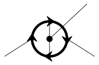
 $=$  $\equiv 0\,,$  $\equiv 0\,,$ (7.62)

 $\equiv 0\,,$  $\equiv 0\,,$ (7.63)

 $\equiv 0\,,$  $\equiv 0\,.$ (7.64)

The above identities hold for any antisymmetric 3-index tensor; in particular, they hold for the Lie algebra structure constants iC_{ijk}. They are proven by mapping a diagram into itself by index transpositions. For example, interchange of the top and bottom vertices in (7.61) maps the diagram into itself, but with the $(-1)^3$ factor.

From the Lie algebra (4.45) it also follows that for any irreducible rep we have

 $= 0\,,$  $= 0\,.$ (7.65)

7.9 DYNKIN LABELS

It is standard to identify a rep of a simple group of rank r by its Dynkin labels, a set of r integers $(a_1 a_2 \ldots a_r)$ assigned to the simple roots of the group by the Dynkin diagrams. The Dynkin diagrams, table 7.7, are the most concise summary of the

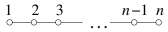
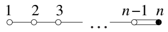
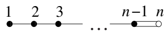
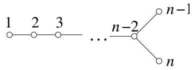
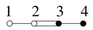
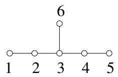
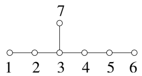
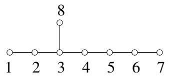
A_r		$\mathrm{SU}(n+1)$
B_r		$\mathrm{SO}(2n+1)$
C_r		$\mathrm{Sp}(2n)$
D_r		$\mathrm{SO}(2n)$
G_2		
F_4		
E_6		
E_7		
E_8		

Table 7.7 Dynkin diagrams for the simple Lie groups.

Cartan-Killing construction of semi-simple Lie algebras. We list them here only to facilitate the identification of the reps and do not attempt to derive or explain them. Dynkin's canonical construction is described in Slansky's review [145]. In this monograph, we emphasize the tensorial techniques for constructing reps. However, in order to help the reader connect the two approaches, we will state the correspondence between the tensor reps (identified by the Young tableaux) and the canonical reps (identified by the Dynkin labels) for each group separately, in the appropriate chapters.

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Chapter Eight

Group integrals

In this chapter we discuss evaluation of group integrals of form

$$\int dg G_a^b G_c^d \ldots G_f^e G_h^g \,, \tag{8.1}$$

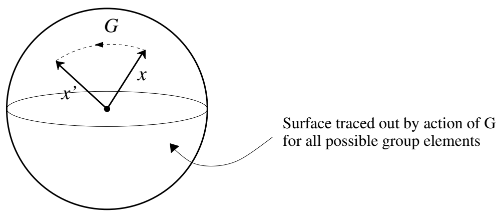
where G_a^b is the $[n\times n]$ defining matrix rep of $g \in G_c$ and the integration is over the entire range of g. As always, we assume that G_c is a compact Lie group, and G_a^b is unitary. Such integrals are of import for certain quantum field theory calculations, and the chapter should probably be skipped by a reader not interested in such applications.

The integral (8.1) is defined by two rules:

1. Normalization:

$$\int dg = 1 \tag{8.2}$$

2. How do we define $\int dg G_a^b$? The action of $g \in G_c$ is to rotate a vector x_a into $x'_a = G_a^{a\,b} x_b$



The averaging smears x in all directions, hence the second integration rule

$$\int dg G_a^b = 0\,, \qquad G \text{ is a } \textit{non-trivial} \text{ rep of } g\,, \tag{8.3}$$

simply states that the average of a vector is zero.

A rep is trivial if $G = 1$ for all group elements g. In this case no averaging is taking place, and the first integration rule (8.2) applies.

What happens if we average a pair of vectors x, y? There is no reason why a pair should average to zero; for example, we know that $|x|^2 = \sum_a x_a x_a^* = x_a x^a$ is invariant (we are considering only unitary reps), so it cannot have a vanishing average. Therefore, in general

$$\int dg G_a^b G_d^c \neq 0\,. \tag{8.4}$$

To get a feeling of what the right-hand side looks like, let us work out a few examples:

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8.1 GROUP INTEGRALS FOR ARBITRARY REPS

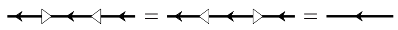
Let G_a^b be the defining $[n\times n]$ matrix rep of $SU(n)$. The defining rep is non-trivial, so it averages to zero by (8.3). The first non-vanishing average is the integral over $G^\dagger$. $G^\dagger$ is the matrix rep of the action of g on the conjugate vector space, which we write as (3.9)

$$G_b^a = (G^\dagger)_b^a\,.$$

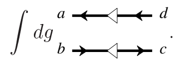
As we shall soon have to face a lot of indices, we immediately resort to birdtracks. In the birdtracks notation of sect. 4.1


(8.5)

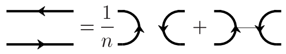
For G the arrows and the triangle point the same way, while for $G^\dagger$ they point the opposite way. Unitarity $G^\dagger G = 1$ is given by


(8.6)

In this notation, the $GG^\dagger$ integral to be evaluated is

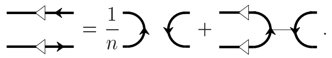

(8.7)

As in the $SU(n)$ example of sect. 2.2, the $V \otimes \overline{V}$ tensors decompose into the singlet and the adjoint rep



$$\delta_a^d \delta_c^b = \frac{1}{n}\delta_a^b \delta_c^d + \frac{1}{a}\left(T_i\right)_a^b \left(T_i\right)_c^d\,. \tag{8.8}$$

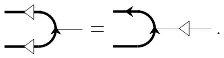
We multiply (8.7) with the above decomposition of the identity. The unitarity relation (8.7) eliminates G's from the singlet:


(8.9)

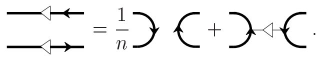
The generators T_i are invariant (see (4.45))

$$(T_i)_b^a = G_{a'}^a G_b^{b'} G_{ii'} \left(T_{i'}\right)_{b'}^{a'}\,, \tag{8.10}$$

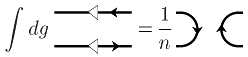
where G_{ij} is the adjoint $[N\times N]$ matrix rep of $g \in G_c$. Multiplying by G_{ii}^{-1}, we obtain


(8.11)

Hence, the pair $GG^\dagger$ in the defining rep can be traded in for a single G in the adjoint rep

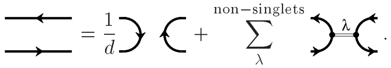

(8.12)

The adjoint rep G_{ij} is non-trivial, so it gets averaged to zero by (8.3). Only the singlet survives



$$\int dg\, G_a^d G_c^b = \frac{1}{n}\delta_c^d \delta_a^b\,. \qquad (8.13)$$

New let G be any irreducible $[d \times d]$ rep. Irreducibility means that any invariant matrix A_b^a is proportional to δ_b^a - otherwise one could use A to construct projection operators of sect. 3.4 and decompose the d-dimensional rep. As the only bilinear type invariant is δ_b^a, the Clebsch-Gordan series contains one and only one singlet



(8.14)

Only the singlet survives the group averaging, and (8.13) is true for any $[d \times d]$ *irreducible* rep (with $n \to d$). If we take G_α^β and G_d^c in inequivalent reps λ, μ (there is no matrix K such that $G^{(\lambda)} = KG^{(\mu)}K^{-1}$ for any $g \in G_c$), then there is no way of forming a singlet, and

$$\int dg G_a^{(\lambda)d} G_\alpha^{(\mu)\beta} = 0 \qquad \text{if} \qquad \lambda \neq \mu. \qquad (8.15)$$

What happens if G is a reducible rep? In the compact index notation of sect. 3.1.4, the group integral (8.1) that we want to evaluate is given by

$$I_\alpha^\beta = \int dg G_\alpha^\beta. \qquad (8.16)$$

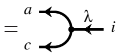
A reducible rep can be expanded in a Clebsch-Gordan series (3.57)

$$I = \sum_\lambda C_\lambda^\dagger \int dg G_\lambda C_\lambda\,. \qquad (8.17)$$

By the second integration rule (8.3), all non-singlet reps average to zero, and one is left with a sum over singlet projection operators

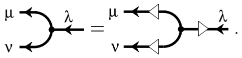
$$\int dg G = \sum_{\text{singlets}} C_\lambda^\dagger C_\lambda = \sum_{\text{singlets}} P_\lambda\,. \qquad (8.18)$$

Group integration amounts to projecting out all singlets in a given Kronecker product. We now flesh out the logic that led to (8.18) with a few details. For concreteness, consider the Clebsch-Gordan series (5.8) for $\mu \times \nu = \sum \lambda$. Each clebsch

$(C_\lambda)^i_{ac} =$  (8.19)

is an invariant tensor (see (4.38)):

$$C^i_{ac,} = G_a^{a'} G_c^{c'} G^i_{i'} C^{i'}_{a'c'},$$



(8.20)

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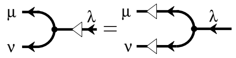
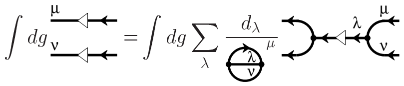
Multiplying with $G^{(\lambda)}$ from left, we obtain the rule for the "propagation" of g through the "vertix" C:



$$C^{i'}_{ac,} G^i_{i'} = G_a^{a'} G_c^{c'} C_{a'c'},{}^i\,. \qquad (8.21)$$

In this way, $G^{(\mu)}G^{(\nu)}$ can be written as a Clebsch-Gordan series, each term with a single matrix $G^{(\lambda)}$ (see (5.8)):



$$= \sum_\lambda (C^\lambda)^i_{ab} (C_\lambda)_j^{cd} \int dg G^{(\lambda)}{}_i^j\,. \qquad (8.22)$$

Clebsches are invariant tensors, so they are untouched by group integration. Integral over $G^{(\mu)}G^{(\nu)}$ reduces to clebsches times integrals

$$\int dg G^{(\lambda)}{}_i^j = \begin{cases} 1 \text{ for } \lambda \text{ singlet} \\ 0 \text{ for } \lambda \text{ non-singlet} \end{cases}\,. \qquad (8.23)$$

Non-trivial reps average to zero, yielding (8.18). We have gone into considerable detail in deriving (8.22) in order to motivate the sum-over-the-singlets projection operators rule (8.18). Clebsches were used in the above derivations for purely pedagogical reasons; all that is actually needed are the singlet projection operators.

8.2 CHARACTERS

Physics calculations (such as lattice gauge theories) often involve group invariant quantities formed by contracting G with invariant tensors. Such invariants are of the form $\mathrm{tr}\,(hG) = h_b^a G_a^b$, where h stands for any invariant tensor. The trace of an irreducible $[d \times d]$ matrix rep λ of g is called the *character* of the rep:

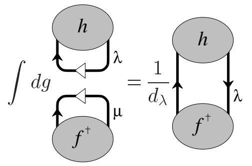
$$\chi_\lambda(g) = \mathrm{tr}\,_\lambda G = G_a^a. \qquad (8.24)$$

The character of the conjugate rep is

$$\chi^\lambda(g) = \chi_\lambda(g)* = \mathrm{tr}\, G^\dagger = (G^\dagger)_a^a. \qquad (8.25)$$

Contracting (8.14) with two arbitrary invariant $[d \times d]$ matrices h_d^a and $(f^\dagger)_b^c$, we obtain the *character orthonormality relation*

$$\int dg \chi_\lambda(hg)\chi^\mu(gf) = \delta_\lambda^\mu \frac{1}{d_\lambda}\chi_\lambda(hg^\dagger) \qquad (8.26)$$



$\left(\begin{array}{l}\lambda, \mu \text{ irreducible} \\ \text{reps}\end{array}\right)$.

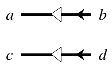
The character orthonormality tells us, that if two group variant quantities share a $GG^\dagger$ pair, the group averaging sews them into a single group invariant quantity. The replacement of G_a^b by the trace $\chi_\lambda(h^\dagger g)$ does not mean that any of the tensor index structure is lost; G_a^b can be recovered by differentiating

$$G_a^b = \frac{d}{dh_b^a}\chi_\lambda(h^\dagger g)\,. \tag{8.27}$$

The birtracks and the characters are two equivalent notations for evaluating group integrals.

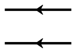
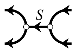
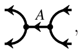
8.3 EXAMPLES OF GROUP INTEGRALS

We will illustrate (8.18) by two examples: $SU(n)$ integrals over GG and $GGG^\dagger G^\dagger$. A product of two $G's$ is drawn as

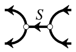
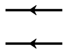
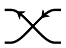
$$G_a^b G_c^d = \tag{8.28}$$


$G's$ are acting on $\otimes V^2$ tensor space which is decomposable by (9.4) into the symmetric and the antisymmetric subspace

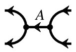
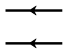
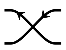
$$\delta_a^b\delta_c^d = (P_s)_{ac},{}^{db} + (P_A)_{ac},{}^{db}$$

 $=$  $+$  , (8.29)

$$(P_s)_{ac},{}^{db} = \frac{1}{2}\left(\delta_a^b\delta_c^d + \delta_a^d\delta_c^b\right)$$

 $= \frac{1}{2}\Big\{$  $+$  $\Big\}$ (8.30)

$$(P_A)_{ac},{}^{db} = \frac{1}{2}\left(\delta_a^b\delta_c^d + \delta_a^d\delta_c^b\right)$$

 $= \frac{1}{2}\Big\{$  $-$  $\Big\}$

$$d_s = \frac{n(n+1)}{2}\,, \qquad d_A = \frac{n(n-1)}{2}\,. \tag{8.31}$$

The transposition of indices b and d is explained in sect. 4.1; it ensures a simple correspondence between tensors and birdtracks.

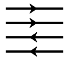
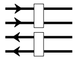
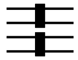
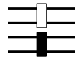
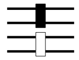
For $SU(2)$ the antisymmetric subspace has dimension $d_A = 1$. We shall return to this case in sect. 15.1. For $n \geq 3$, both subspaces are non-singlets, and by the second integration rule

$$SU(n): \int dg G_a^b G_c^d = 0\,, n > 2\,. \tag{8.32}$$

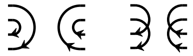
As the second example, we take the group integral over $GGG^\dagger G^\dagger$.

This rep acts on $V^2 \otimes \overline{V}^2$ tensor space. There are various ways of constructing the singlet projectors; we shall give two.

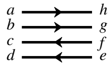
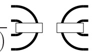
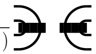
We can treat the $V^2 \otimes \overline{V}^2$ space as a Kronecker product of spaces $\otimes V^2$ and $\otimes\overline{V}^2$. We first reduce the particle and antiparticle spaces separately by (8.29)

 $=$  $+$  $+$  $+$  . (8.33)

The only invariant tensors that can project singlets out of this space (for $n \geq 3$) are index contraction with no intermediate lines;

 . (8.34)

Contracted with the last two reps in (8.33), they yield zero. Only the first two reps yield singlets

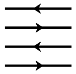
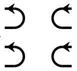
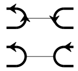
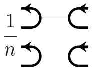
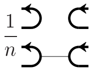
 $\Rightarrow \frac{2}{n(n+1)}$  $+ \frac{2}{n(n-1)}$  . (8.35)

The projector normalization factors are the dimensions of the associated reps (3.21). The $GGG^\dagger G^\dagger$ group integral written out in tensor notation is

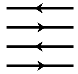
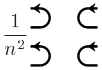
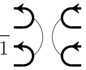
$$\begin{aligned}\int dg G_h^a G_g^b G_c^i G_d^e = &\frac{1}{2n(n+1)}\left(\delta_d^a\delta_c^b + \delta_c^a\delta_d^b\right)\left(\delta_h^e\delta_g^f + \delta_g^e\delta_h^f\right)\\ &+\frac{1}{2n(n-1)}\left(\delta_d^a\delta_c^b - \delta_c^a\delta_d^b\right)\left(\delta_h^e\delta_g^f - \delta_g^e\delta_h^f\right)\,.\end{aligned} \tag{8.36}$$

We have obtained this result by first reducing $\otimes V^2$ and $\otimes\overline{V}^2$. What happens if we reduce $V^2 \otimes \overline{V}^2$ as $(V \otimes \overline{V})^2$?

We first decompose the two $V \otimes \overline{V}$ tensor subspaces into singlets and adjoint reps (see sect. 2.2):

 $= \frac{1}{n^2}$  $+$  $+ \frac{1}{n}$  $+ \frac{1}{n}$  . (8.37)

The two cross terms with one intermediate adjoint line cannot be reduced further. The 2-index adjoint intermediate state contains only one singlet in the Clebsch-Gordan series (15.25), so that the final result [33] is

 $= \frac{1}{n^2}$  $+ \frac{1}{n^2-1}$  . (8.38)

It can be checked, by substituting adjoint rep projection operators (9.45), that this is the same combination of Kronecker deltas as (8.36).

To summarize, the projection operators constructed in this monograph are all that is needed for evaluation of group integrals; the group integral for an arbitrary rep is given by the sum over all singlets (8.18) contained in the rep.

Chapter Nine

Unitary groups

(P. Cvitanović, H. Elvang, and A. D. Kennedy)

$U(n)$ is the group of all transformations which leave invariant the norm $\overline{q}q = \delta_b^a q^b q_a$. of a complex vector q. For $U(n)$ there are no other invariant tensors beyond those constructed of products of Kronecker deltas. They can be used to decompose the tensor reps of $U(n)$. For purely covariant or contravariant tensors, the symmetric group can be used to construct the Young projection operators. In sects. 9.1–9.2 we show how to do this for 2- and 3-index tensors by constructing the appropriate characteristic equations. For tensors with more indices it is easier to construct the Young projection operators directly from the Young tableaux. We use the projection operators so constructed to evaluate characters and 3-j coefficients of $U(n)$.

For mixed tensors reduction also involves index contractions and the symmetric group methods alone do not suffice. In sects. 9.8–9.10 the mixed $U(n)$ tensors are decomposed by the projection operator techniques introduced in chapter 3.

9.1 TWO-INDEX TENSORS

Consider 2-index tensors $q^{(1)} \otimes q^{(2)} \in V^2$. According to (6.1), all permutations are represented by invariant matrices. Here there are only two permutations, the identity and the flip (6.2)

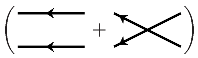
$\sigma =$  .

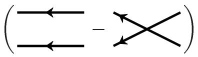
The flip satisfies

$\sigma^2 =$  $= \mathbf{1}$,

$$(\sigma + \mathbf{1})(\sigma - \mathbf{1}) = 0 . \qquad (9.1)$$

Hence, the roots are $\lambda_1 = 1$, $\lambda_2 = -1$, and the corresponding projection operators (3.45) are

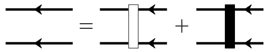
$P_1 = \frac{\sigma - (-1)\mathbf{1}}{1-(-1)} = \frac{1}{2}(\mathbf{1} + \sigma) = \frac{1}{2}$  , (9.2)

$P_2 = \frac{\sigma - \mathbf{1}}{-1-1} = \frac{1}{2}(\mathbf{1} - \sigma) = \frac{1}{2}$  . (9.3)

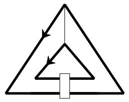
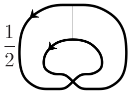
We recognize the symmetrization, antisymmetrization operators (6.4), (6.15); $P_1 = S$, $P_2 = A$, with subspace dimensions $d_1 = n(n+1)/2$, $d_2 = n(n-1)/2$. In other

words, under general linear transformations the symmetric and the antisymmetric parts of a tensor x_{ab} transform separately:

$$x = Sx + Ax ,$$
$$x_{ab} = \frac{1}{2}(x_{ab} + x_{ba}) + \frac{1}{2}(x_{ab} - x_{ba})$$

 . (9.4)

The Dynkin indices for the two reps follow by (7.28) from $6j's$:

 $= \frac{1}{2}(0) + \frac{1}{2}$  $= \frac{N}{2}$

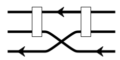
$$\ell_1 = \frac{2\ell}{n} \cdot d_1 + \frac{2\ell}{N} \cdot \frac{N}{2}$$
$$= \ell(n+2) . \qquad (9.5)$$

Substituting the defining rep Dynkin index $\ell^{-1} = C_A = 2n$, computed in sect. 2.2, we obtain the two Dynkin indices

$$\ell_1 = \frac{n+2}{2n} , \qquad \ell_2 = \frac{n-2}{2n} . \qquad (9.6)$$

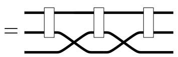
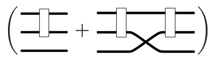
9.2 THREE-INDEX TENSORS

3-index tensors can be reduced to irreducible subspaces by adding the third index to each of the 2-index subspaces, the symmetric and the antisymmetric. The results of this section are summarized in table 9.1 and table 9.3. We mix the third index into the symmetric 2-index subspace using the invariant matrix

$Q = S_{12}\sigma_{(23)}S_{12} =$  . (9.7)

Here projection operators S_{12} ensure the restriction to the 2-index symmetric subspace, and the transposition $\sigma_{(23)}$ mixes in the third index. To find the characteristic equation for Q, we compute Q^2:

$$Q^2 = S_{12}\sigma_{(23)}S_{12}\sigma_{(23)}S_{12} = \frac{1}{2}\left(S_{12} + S_{12}\sigma_{(23)}S_{12}\right) = \frac{1}{2}S_{12} + \frac{1}{2}Q$$

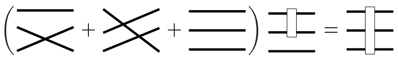
$=$  $= \frac{1}{2}$  .

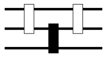
Hence, Q satisfies

$$(Q-1)(Q+1/2)S_{12} = 0 , \qquad (9.8)$$

and the corresponding projection operators (3.45) are

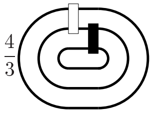
$$P_1 = \frac{Q + \frac{1}{2}\mathbf{1}}{1 + \frac{1}{2}} S_{12} = \frac{1}{3}\left(\sigma_{(23)} + \sigma_{(123)} + \mathbf{1}\right) S_{12} = S$$

$$= \frac{1}{3}$$  (9.9)

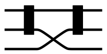
$$P_2 = \frac{Q - \mathbf{1}}{-\frac{1}{2} - 1} S_{12} = \frac{4}{3} S_{12} A_{23} S_{12} = \frac{4}{3}$$  . (9.10)

Hence, the symmetric 2-index subspace combines with the third index into a symmetric 3-index subspace (6.13) and a mixed symmetry subspace with dimensions

$$d_1 = \mathrm{tr}\, P_1 = \frac{n(n+1)(n+2)}{3!} \qquad (9.11)$$

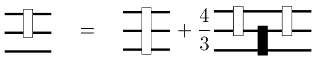
$$d_2 = \mathrm{tr}\, P_2 = \frac{4}{3}$$  $$= \frac{n(n^2-1)}{3} . \qquad (9.12)$$

The antisymmetric 2-index subspace can be treated in the same way using invariant matrix

$$Q = A_{12} \sigma_{(23)} A_{12} =$$  . (9.13)

The resulting projection operators for the antisymmetric and mixed symmetry 3-index tensors are given in table 9.1. Symmetries of the subspace are indicated by the corresponding Young tableaux, table 9.2. For example, we have just constructed

$$\boxed{1}\boxed{2} \otimes \boxed{3} = \boxed{1}\boxed{2}\boxed{3} \oplus \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}$$



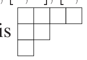
$$\frac{n^2(n+1)}{2} = \frac{n(n+1)(n+2)}{3!} + \frac{n(n^2-1)}{3} . \qquad (9.14)$$

9.3 YOUNG TABLEAUX

As we have seen in the above examples, the projection operators for 2-index and 3-index tensors can be constructed using the characteristic equations. This, however, becomes cumbersome when applied to tensors with more than 3 indices. We now show how to construct Young projection operators for the irreducible representations of $U(n)$ directly from the Young tableaux.

9.3.1 Definitions

Partition k boxes into D subsets, so that the mth subset contains λ_i boxes. Order the partition so the set $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_D]$ fulfills $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_D \geq 1$ and $\sum_{i=1}^{D} \lambda_i = k$. The diagram obtained by drawing the D rows of boxes on top of each other, left aligned, starting with λ_1, is called a *Young diagram* Y.

EXAMPLES: For $k = 4$ the ordered partitions for $k = 4$ are $[4]$, $[3,1]$, $[2,2]$, $[2,1,1]$ and $[1,1,1,1]$. For the $k = 7$ partition $[4,2,1]$ the Young diagram is  and for the $k = 3$ partition $[1,1,1]$ it is $\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array}$.

A box in a Young diagram can be assigned a coordinate (i,j) such that $\mathrm{Y} = \{(i,j) \in \mathbb{Z}^2 | 1 \leq j \leq \lambda_i\}$. Here i label the rows and j the columns.

Inserting a number from the set $\{1, \ldots, n\}$ into every box of a Young diagram Y_λ in such a way that numbers increase when reading a column from top to bottom, and numbers do not decrease when reading a row from left to right yields a *Young tableau* Y_a. The subscript a labels different tableaux derived from a given Young diagram, that is different admissible ways of inserting the numbers into the boxes. Denoting the number in the (i,j)th box by $\tau_a(i,j)$, we have

$$\begin{aligned} \mathrm{Y}_a = \{(\tau_a(i,j)) \in \{1, \ldots, n\}^k \mid (i,j) \in \mathrm{Y}, \\ \tau_a(i, j+1) \geq \tau_a(i,j), \\ \tau_a(i+1, j) > \tau_a(i,j)\} \end{aligned}$$

A Young tableau with numbers inserted as above is called a *standard arrangement*. The monotonically ordered arrangement

$$\begin{aligned} \mathrm{Y}_a = \{(\tau_a(i,j)) \in \{1, \ldots, k\} \mid (i,j) \in \mathrm{Y}, \\ \tau_a(i, j+1) > \tau_a(i,j), \\ \tau_a(i+1, j) > \tau_a(i,j)\} \end{aligned}$$

is called a k*-standard arrangement*.

In the following, we denote by Young diagram Y a box diagram without numbers, and by Young tableaux Y_a a diagram filled with a standard arrangement. Often we simplify the notation by using Y, Z, …to denote *both* Young diagrams and Young tableaux.

The *transpose* diagram Y^t is obtained from Y by interchanging rows and columns. For example, the transpose of $[3,1]$ is $[2,1,1]$.

An alternative labelling of a Young diagram is to list the number b_m of columns with m boxes as $(b_1 b_2 \ldots)$. Having k boxes we must have $\sum_{m=1}^{k} m b_m = k$. As an example, we see that $[4,2,1]$ and $(21100\ldots)$ label the same Young diagram. Similarly for $[2,2]$ and $(020\ldots)$. This notation is handy when considering Dynkin labels.

9.3.2 $SU(n)$ Young tableaux

We now show that a Young tableau with no more than n rows corresponds to an irreducible rep of $SU(n)$.

A k-index tensor is represented by a Young diagram with k boxes — one may think of this as a k-particle state. For $SU(n)$ there are n 1-particle states available, and the irreducible k-particle states correspond to a Young tableaux obtained by inserting the numbers $1, \ldots, n$ into the k boxes of the Young diagrams. Boxes in a row correspond to indices that are symmetric under interchanges (symmetric multiparticle states), and boxes in a column correspond to indices antisymmetric under interchanges (antisymmetric multiparticle states).

Consider the reduction of a 2-particle state, that is a 2-index tensor, into a symmetric and an antisymmetric state (9.4). Using Young diagrams we would write this as

$$\ydiagram{1} \otimes \ydiagram{1} = \ydiagram{2} \oplus \ydiagram{1,1} \,. \tag{9.15}$$

For the $n = 2$ case the Young tableaux of SU(2) are:

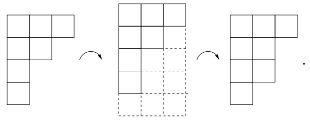
$$\begin{ytableau} 1 & 1 \end{ytableau}\,, \quad \begin{ytableau} 1 & 2 \end{ytableau}\,, \quad \begin{ytableau} 2 & 2 \end{ytableau} \quad \text{and} \quad \begin{ytableau} 1 \\ 2 \end{ytableau}\,.$$

The dimension of an irreducible rep of $SU(n)$ is found by counting the number of standard arrangements. Thus for SU(2) the symmetric state is 3 dimensional, whereas the antisymmetric state is 1 dimensional, in agreement with the formulas (6.4) and (6.15) for the dimensions of the symmetry operators. In sect. 9.4.1 we shall state and prove the dimension formula for a general irreducible $U(n)$ rep.

A rep of $SU(n)$, or A_{n-1} in the Cartan classification, table 7.7, is characterized by n-1 *Dynkin labels* $a_1 a_2 \ldots a_{n-1}$. The corresponding Young tableau (defined in sect. 9.3.1) is given by $(a_1 a_2 \ldots a_{n-1} 00 \ldots)$. For example, for $SU(3)$

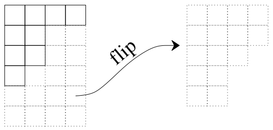
$$\begin{array}{llll} (10) = \ydiagram{1} = 3 & & (20) = \ydiagram{2} = 6 & \\ (01) = \ydiagram{1,1} = \overline{3} & & (02) = \ydiagram{2,2} = \overline{6} & \\ (11) = \ydiagram{2,1} = 8 & & (21) = \ydiagram{3,1} = 15 \,. & \end{array} \tag{9.16}$$

For $SU(n)$ columns cannot contain more than n boxes, as it is impossible to antisymmetrize more than n labels. Columns of n boxes can be contracted away by means of the Levi-Civita tensor (6.27). Hence, the highest column is of height n-1, which is also the rank of $SU(n)$. Furthermore, for $SU(n)$ a column with k boxes (antisymmetrization of covariant k indices) can be converted by contraction with the Levi-Civita tensor into a column of $(n$-$k)$ boxes (corresponding to $(n$-$k)$ contravariant indices). This operation associates with each tableau a conjugate rep. Thus, the *conjugate* of a $SU(n)$ Young diagram Y is constructed from the missing pieces needed to complete the rectangle of n rows:



(9.17)

That is, add squares below the diagram of Y such that the resulting figure is a rectangle with height n and width of the top row in Y. Remove the squares corresponding to Y and rotate the rest by 180 degrees. The result is the conjugate diagram of Y. For example, for $SU(6)$, rep (20110)



(9.18)

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has as its conjugate rep (01102). In general, the $SU(n)$ reps $(b_1 b_2 \ldots b_{n-1})$ and $(b_{n-1} \ldots b_2 b_1)$ are conjugate. For example, if $(10 \ldots 0)$ stands for the defining rep, then its conjugate is represented by $(00 \ldots 01)$, *ie.* a column of n-1 boxes.

We prefer to keep the conjugate reps conjugate, rather than replacing them by columns of $(n$-$1)$ defining reps, as this will give us $SU(n)$ expressions valid for any n.

9.3.3 Reduction of direct products

We now state the rules for reduction of direct products such as (9.15) in terms of Young diagrams:

Draw the two diagrams next to one another and place in each box of the second diagram an a_i, $i = 1, \ldots, k$, such that the boxes in the first row all have a_1 in them, second row boxes have a_2 in them *etc*. The boxes of the second diagram are now added to the first diagram to create new diagrams in accordance to the rules

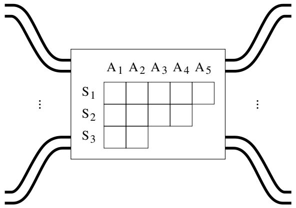
1. Each diagram must be a Young diagram.
2. The number of boxes in the new diagram must be equal to the sum of the number of boxes in the original two diagrams.
3. For SU(n) no diagram has more than n rows.
4. Making a journey through the diagram starting with the top row and entering each row from the right, at any point the number of a_i's encountered in any of the attached boxes must not exceed the number of previously encountered a_{i-1}'s.
5. The numbers must not increase when reading across a row from left to right.
6. The numbers must decrease when reading a column from top to bottom.

The rules 4-6 ensure that states which were previously symmetrized are not antisymmetrized in the product and vice versa and to avoid counting the same state twice.

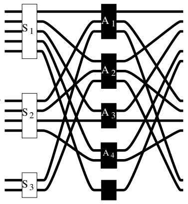
9.4 Young projection operators

Given a Young tableau Y of $U(n)$ with an k-standard arrangement we construct the corresponding Young projection operator P_{Y} in birdtrack notation by identifying each box in the diagram with a directed line. The operator P_{Y} is a block of symmetrizers to the left of a block of antisymmetrizers, all imposed on the n lines. The blocks of symmetry operators are dictated by the Young *diagram* whereas the attachment of lines to these operators follows from the k-standard arrangement.

For a Young diagram Y with s rows and t columns we refer to the rows as S_1, S_2, …,S_s and to the columns as A_1, A_2, …,A_t. Each symmetry operator in P_{Y} is associated to a row/column in Y, hence we label a symmetry operator after the corresponding row/column,

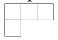


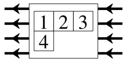
$= \alpha_Y$

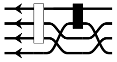


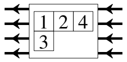
. (9.19)

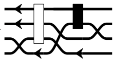
We denote by $|S_i|$ or $|A_i|$ the *length* of a row or column, respectively, that is the number of boxes it contains. Thus $|A_i|$ also denotes the number of lines entering the antisymmetrizer A_i. In the above example we have $|S_1| = 5$, and $|A_2| = 3$, *etc*.

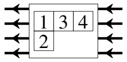
An example of the construction of the Young projection operators: The Young diagram  tells us to use one symmetrizer of length three, one of length one, one antisymmetrizer of length two, and two of length one. There are three distinct k-standard arrangements, each corresponding to a projection operator

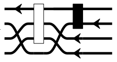


$= \alpha_Y$  (9.20)



$= \alpha_Y$  (9.21)



$= \alpha_Y$  , (9.22)

where α_Y is a normalization constant. We use the convention, that if the lines pass straight through the symmetry operators, they appear in the same order as they entered. More examples of Young projection operators are given in sect. 9.5. The normalization is given by

$$\alpha_Y = \frac{\prod_{i=1}^{s} |S_i|! \prod_{j=1}^{t} |A_j|!}{|Y|}, \tag{9.23}$$

where $|Y|$ is a combinatoric number calculated by the following hook rule. For each box of the Young diagram Y write the number of boxes below and to the left of the box (including the box itself — once). Then $|Y|$ is the product of the numbers in all the boxes. For instance,

$$Y = \begin{array}{|c|c|c|c|} \hline 6 & 5 & 3 & 1 \\ \hline 4 & 3 & 1 \\ \cline{1-3} 2 & 1 \\ \cline{1-2} \end{array} \tag{9.24}$$

has $|Y| = 6! \cdot 3$. We prove that this is the correct normalization in appendix B. The normalization only depends on the Young diagram, not the particular tableau.

For multidimensional irreducible reps the Young projection operators constructed as above, will generally be different from the ones constructed from characteristic equations, see sects. 9.1–9.2, but the difference amounts to a choice of basis, so they are equivalent.

We prove in appendix B that the above construction indeed yields well-defined projection operators. Some of the properties of the Young projection operators:

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- The Young projection operators are indeed *projection operators*, $P_Y^2 = P_Y$.
- The Young projection operators are *orthogonal*: If Y and Z are two different k-standard arrangement, then $P_Y P_Z = 0 = P_Z P_Y$.
- For a given k the Young projection operators constitute a complete set such that $\mathbf{1} = \sum P_Y$, where the sum is over all k-standard arrangements Y with k boxes, and $\mathbf{1}$ is the $[k \times k]$ unit matrix.

The dimension $d_Y = \mathrm{tr}\, P_Y$ of a Young projection operator P_Y can be calculated directly by tracing P_Y and expanding it using (6.10) and (6.19). In practice, this is unnecessarily laborious. Instead, we offer two simple ways of computing the dimension of an irreducible rep from its Young diagram.

9.4.1 A dimension formula

Let $f_Y(n)$ be the polynomial in n obtained from the Young diagram Y by multiplying the numbers written in the boxes of Y, according to the following rules:

1. The upper left box contains an n.
2. The numbers in a row increases by one when reading from left to right.
3. The numbers in a column decrease by one when reading from top to bottom.

Hence, if k is the number of boxes in Y, $f_Y(n)$ is a polynomial in n of degree k.

For $U(n)$ the dimension of the irreducible rep, labelled by the Young diagram Y, is

$$d_Y = \frac{f_Y(n)}{|Y|}. \tag{9.25}$$

EXAMPLE: With Y = [4,2,1], we have

$$f_Y(n) = \begin{array}{|c|c|c|c|} \hline n & n{+}1 & n{+}2 & n{+}3 \\ \hline n{-}1 & n \\ \cline{1-2} n{-}2 \\ \cline{1-1} \end{array} = n^2(n^2-1)^2(n^2-4)(n+3),$$

$$|Y| = \begin{array}{|c|c|c|c|} \hline 6 & 4 & 2 & 1 \\ \hline 3 & 1 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array} = 144, \tag{9.26}$$

hence,

$$d_Y = \frac{n^2(n^2-1)^2(n^2-4)(n+3)}{144}. \tag{9.27}$$

. This dimension formula is derived in appendix B. Next we give an intuitive interpretation of what this formula means.

9.4.2 Dimension as the number of strand colorings

The dimension of a Young projection operator P_Y of $SU(n)$ can be calculated by counting the number of distinct ways, in which the trace diagram of a Young projection operator can be colored.

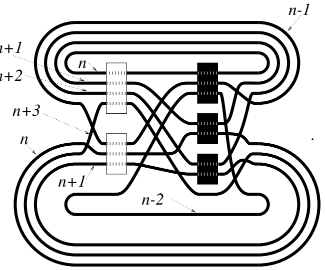
Draw the trace of the Young projection operator. Each line is strand, a closed path which we draw as passing straight through the symmetry operators. Order the paths in accordance to the k-standard arrangement (see example). The lines are colored in this order. Having n colors we can color the first line in n different ways.

Rule 1: If a path, which could be colored in k ways, enters an antisymmetrizer, the lines below it can be colored in $k-1$, $k-2$, …ways.

Rule 2: If a path, which could be colored in k ways, enters a symmetrizer, the lines below it can be colored in $k+1$, $k+2$, …ways.

Label each path with the number of ways it can be colored. The number of ways to color the trace diagram is the product of all the factors obtained above; but this is simply $f_Y(n)$ defined in sect. 9.4.1. An example:

$$d_Y = \frac{f_Y(n)}{|Y|} = \mathrm{tr}\,\begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 4&5&7\\\cline{1-3} 8\\\cline{1-1}\end{array} = \frac{1}{|Y|}$$



. (9.28)

9.5 REDUCTION OF TENSOR PRODUCTS

We now apply the rules for decomposition of direct products of Young diagrams/tableaux to several explicit examples. We use the tableaux to compute the dimensions and construct the Young projection operators. We have already treated the decomposition of the 2-index tensor into the symmetric and the anti-symmetric tensors, but we shall reconsider the 3-index tensor, since the projection operators will be different from those derived from the characteristic equations in sect. 9.2.

9.5.1 Three- and four-index tensors

According to the rules in sect. 9.3.3, the 3-index tensor reduces to

$$\boxed{1}\otimes\boxed{2}\otimes\boxed{3} = \left(\begin{array}{|c|c|}\hline 1&2\\\hline\end{array} \oplus \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right)\otimes\boxed{3} = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} \oplus \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \oplus \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} \quad (9.29)$$

The corresponding dimensions and Young projection operators are given in table 9.3. For simplicity, we neglect the arrows on the lines where this leads to no confusion.

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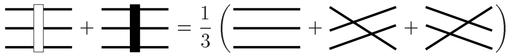
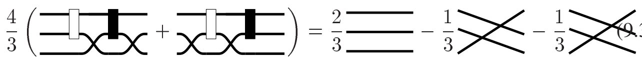
Let us check the completeness by an computation. In the sum of the fully symmetric and the fully antisymmetric tensors all the odd permutations cancel, and we are left with



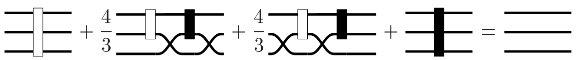
(9.30)

Expanding the two tensors of mixed symmetry, we obtain



(9.31)

Adding (9.30) and (9.31) we get



,(9.32)

verifying the completeness relation.

For 4-index tensors the decomposition is performed as in the 3-index case, resulting in table 9.4.

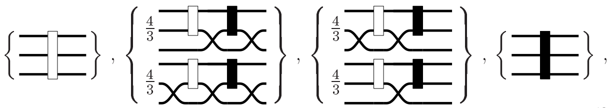
9.5.2 Basis vectors

The Young projection operators as constructed above are also projection operators of the symmetric group S_n. If we let Y be a Young tableau labelling an irreducible rep of S_n, the dimension of the rep is

$$d_Y = \frac{n!}{|Y|}. \quad (9.33)$$

For the 2-index tensors we see that application of the projection operators project any group element to the subspace in question.

For the 3-index tensors the result is not as simple as that, because the S_n rep  is 2-dimensional. Instead, when the 3-index projection operators are applied from the right, the group elements of S_n are projected to the set

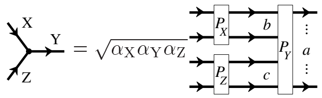


(9.34)

of basis vectors. For higher index tensors there are similar sets of basis vectors. The number of components in each basis vector is the dimension of the projection operator in S_n.

9.6 3-J SYMBOLS

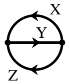
The $SU(n)$ 3-vertex is written



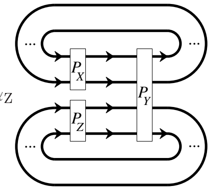
(9.35)

in terms of the Young projection operators P_X, P_Y, and P_Z. If $b+c \neq a$ the vertex vanishes; if $a = b+c$ the vertex might be non-vanishing. The overall normalization is arbitrary, but $\sqrt{\alpha_X \alpha_Y \alpha_Z}$ is a natural choice, see (9.23).

A 3-j consists of two fully contracted 3-vertices. We, therefore, have



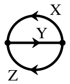
$$= \alpha_X \alpha_Y \alpha_Z \tag{9.36}$$



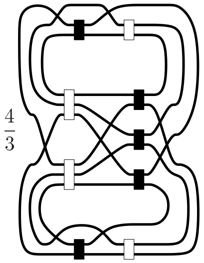
which we write $\mathrm{tr}\,(X \oplus Z) \otimes Y$. As an example, take

$$X = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \quad Y = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline\end{array}\,, \qquad \text{and} \qquad Z = \begin{array}{|c|c|}\hline 4 & 5 \\ \hline 6 \\ \cline{1-1}\end{array}.$$

Then



$$= \frac{4}{3} \cdot 2 \cdot \frac{4}{3}$$



.

For economy of notation, we omit the arrows on the Kronecker delta lines.

9.6.1 Evaluation by direct expansion

The simplest 3-j's to evaluate are $\mathrm{tr}\,(\square \oplus \square) \otimes \square\square$ and $\mathrm{tr}\,(\square \oplus \square) \otimes \begin{array}{c}\square\\[-4pt]\square\end{array}$.

Any $SU(n)$ 3-j may be evaluated by direct expansion of the symmetry operators, but the resulting number of terms grows combinatorially with the total number of boxes in the Young diagram Y, making brute force expansion an unattractive method.

There is a slightly less brutal expansion method. Expanding one symmetry operator may lead to simplifications of the diagram, for instance by using rules such as (6.7), (6.8), (6.17), and (6.18). An example of the application of this method is given in (Elvang).

If Y is a Young diagram with a single row or a single column, it is easily seen that the 3-j $X \otimes Y \otimes Z$ is either 0 or d_Y.

9.6.2 An application of the negative dimension theorem

An $SU(n)$ invariant scalar is a fully contracted object (vacuum bubble) consisting of Kronecker deltas and Levi-Civita symbols. Since there are no external legs, the Levi-Civitas appear only in pairs, making it possible to combine them into antisymmetrizers. In the birdtrack notation, an $SU(n)$ invariant scalar is therefore a vacuum bubble graph built only from symmetrizers and antisymmetrizers.

The negative dimensionality theorem for $SU(n)$ states that for any $SU(n)$ invariant scalar exchanging symmetrizers and antisymmetrizers is equivalent to replacing n by $-n$:

$$\mathrm{SU}(n) = \overline{\mathrm{SU}}(-n)\,, \tag{9.37}$$

where the bar on $\overline{\mathrm{SU}}$ indicates transposition, *ie.* exchange of symmetrizations and antisymmetrizations. The theorem also applies to $U(n)$ invariant scalars, since the only difference between $U(n)$ and $SU(n)$ is the invariance of the Levi-Civita tensor in $SU(n)$. The proof of this theorem is given in chapter 13.

For the dimensions of the Young projection operators we have $d_{Y^t}(n) = d_Y(-n)$ by the negative dimensionality theorem, where Y^t is the transpose of the k-standard arrangement Y; hence, it suffices to compute the dimension once, either for Y or Y^t.

Now for k-standard arrangements X, Y, and Z, compare the diagram of $X^t \otimes Y^t \otimes Z^t$ to that of $X \otimes Y \otimes Z$. The diagrams are related by a reflection in a vertical line, reversal of the arrows on the lines, and interchange of symmetrizers and antisymmetrizers. The first two operations do not change the value of the diagram, hence, the value of $X^t \otimes Y^t \otimes Z^t$ is the value of $X \otimes Y \otimes Z$ with $n \leftrightarrow -n$ (and possibly an overall sign). Hence, it is sufficient to calculate approximately half of all 3-j's.

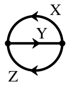
9.6.2.1 Challenge

We have seen that there is a coloring algorithm for the dimensionality of the Young projection operators. Find a coloring algorithm for the 3-j's of $SU(n)$ — *open question.*

9.6.3 A sum rule for 3-j's

Let Y be a k-standard arrangement with k boxes, and let Λ be the set of all k-standard arrangements and Λ_p the set of k-standard arrangements with p boxes. Then

$$\sum_{(X,Z)\in\Lambda}$$

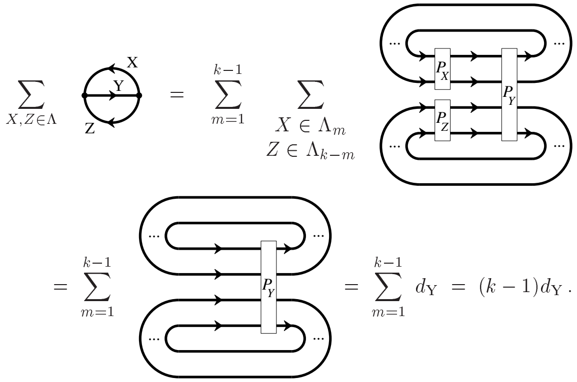


$$= (k-1)\, d_Y. \tag{9.38}$$

First of all, the sum is well-defined, *ie.* finite, because the 3-j is non-vanishing only if the number of boxes in X and Z add up to k, and this only happens for finitely many tableaux.

To prove this, recall that the Young projection operators constitute a complete set,

$\sum_{X\in\Lambda_p} P_X = \mathbf{1}$, where $\mathbf{1}$ is the $[p \times p]$ unit matrix. Hence,



(9.39)

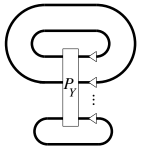
This sum rule offers a cross-check on the individual 3-j calculations.

9.7 CHARACTERS

Now that we have explicit Young projection operators we should be able to compute any $SU(n)$ invariant scalar. As an example, we will evaluate several characters (introduced in sect. 8.2) for $SU(n)$.

Given an irreducible rep, we have the corresponding Young tableau k-standard arrangement Y, which enable us to calculate the character $\chi_Y(M) = \mathrm{tr}_Y M$, where M is a unitary $[n \times n]$ matrix.

Diagrammatically we shall denote M as $M_{ij} = j \longleftarrow i$. Then

$\chi_Y(M) =$  . (9.40)

Expanding the symmetry operators and collecting terms, we find

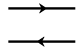
$$\chi_Y(M) = \sum_{m=0}^{k} c_m (\mathrm{tr}\, M)^m \mathrm{tr}\, M^{k-m} \,, \tag{9.41}$$

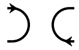
where k is the number of boxes in Y, and the c_m's are coefficients of the expansion.

9.8 MIXED TWO-INDEX TENSORS

As the next example consider mixed tensors $q^{(1)} \otimes \overline{q}^{(2)} \in V \otimes \overline{V}$. The Kronecker delta invariants are the same as in sect. 9.1, but now they are drawn differently (we

are looking at a "cross channel"):

identity: $\mathbf{1} = 1_{a,d}^{b\ c} = \delta_a^c \delta_d^b =$  ,

trace: $T = T_{a,d}^{b\ c} = \delta_a^b \delta_d^c =$  . (9.42)

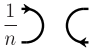
The T matrix satisfies a trivial characteristic equation

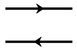
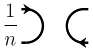
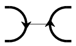
$T^2 =$  $= nT\,,$ (9.43)

with roots $\lambda_1 = 0,\ \lambda_2 = n$;

$$T(T-n) = 0\,.$$

The corresponding projection operators (3.45) are

$P_1 = \frac{1}{n}T = \frac{1}{n}$  , (9.44)

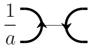
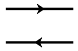
$P_2 = \mathbf{1} - \frac{1}{n}T =$  $- \frac{1}{n}$  $=$  , (9.45)

with dimensions $d_1 = \mathrm{tr}\, P_1 = 1$, $d_2 = \mathrm{tr}\, P_2 = n^2 - 1$. P_2 is the projection operator for the adjoint rep of $SU(n)$. In this way, the invariant matrix T has resolved the space of tensors $x_b^a \in V \otimes \overline{V}$ into

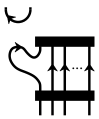
$$\text{singlet:}\quad P_1 x = \frac{1}{n} x_c^c \delta_a^b \,, \tag{9.46}$$

$$\text{traceless part:}\quad P_2 x = x_a^b - \left(\frac{1}{n} x_c^c\right) \delta_a^b \,. \tag{9.47}$$

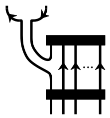
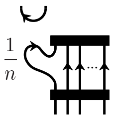
Both projection operators leave δ_b^a invariant, so the generators of the unitary transformations are given by their sum

$U(n):\quad \frac{1}{a}$  $=$  , (9.48)

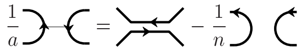
and the dimension of the $U(n)$ adjoint rep is $N = \mathrm{tr}\, P_A = \delta_a^a \delta_b^b = n^2$. If we extend the list of primitive invariants from the Kronecker delta to the Kronecker delta and the Levi-Civita tensor (6.27), the singlet subspace does not satisfy the invariance condition (6.58)

 $\neq 0\,.$

For the traceless subspace (9.45), the invariance condition is

 $- \frac{1}{n}$  $= 0\,.$

This is the same relation as (6.25), as can be shown by expanding the antisymmetrization operator using (6.19), so the invariance condition is satisfied. The adjoint rep is given by

$SU(n):$ 

$$\frac{1}{a}\left(T_i\right)_b^a\left(T_i\right)_c^d=\delta_c^a \delta_b^d-\frac{1}{n} \delta_b^a \delta_c^d . \qquad (9.49)$$

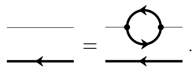
The special unitary group $SU(n)$ is, by definition, the invariance group of the Levi-Civita tensor (hence "special") and the Kronecker delta (hence "unitary"), and its dimension is $N = n^2 - 1$. The defining rep Dynkin index follows from (7.26) and (7.27)

$$\ell^{-1} = 2n \qquad (9.50)$$

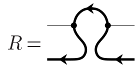
(This was evaluated in the example of sect. 2.2). The Dynkin index for the singlet rep (9.46) vanishes identically, as it does for any singlet rep.

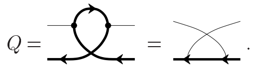
9.9 MIXED DEFINING $\otimes$ ADJOINT TENSORS

In this and the following section we generalize the reduction by invariant matrices to spaces other than the defining rep. Such techniques will be very useful later on, in our construction of the exceptional Lie groups. We consider the defining $\otimes$ adjoint tensor space as a projection from $V \otimes \overline{V}$ space:

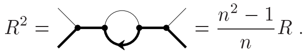
 (9.51)

The following two invariant matrices acting on $V^2 \otimes \overline{V}$ space contract or interchange defining rep indices:

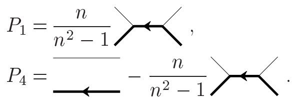
 (9.52)

 (9.53)

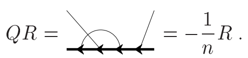
R projects onto the defining space and satisfies characteristic equation

 (9.54)

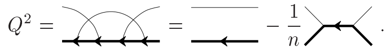
The corresponding projection operators (3.45) are

 (9.55)

Q takes a single eigenvalue on the P_1 subspace

 (9.56)

Q^2 is computed by inserting the adjoint rep projection operator (9.49):

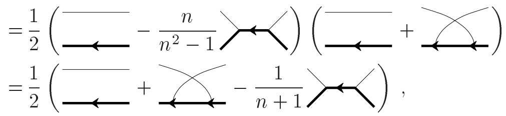
 (9.57)

The projection on the P_4 subspace yields the characteristic equation

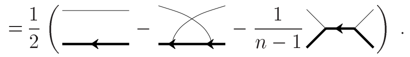
$$P_4(Q^2 - 1) = 0\,, \qquad (9.58)$$

with the associated projection operators

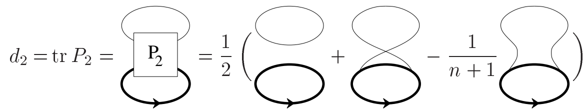
$$P_2 = \frac{1}{2}P_4(1+Q) \qquad (9.59)$$



$$P_3 = \frac{1}{2}P_4(1-Q)$$

 (9.60)

The dimensions of the two subspaces are computed by taking traces of their projection operators:

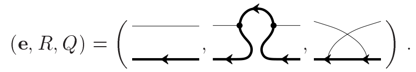


$$= \frac{1}{2}\left(nN + N - \frac{1}{n+1}N\right)\left(n+1-\frac{1}{n+1}\right)$$

$$= \frac{(n-1)n(n+2)}{2} \qquad (9.61)$$

and similarly for d_3. This is tabulated in table 9.5.

Mostly for illustration purposes, let us now perform the same calculation by utilizing the algebra of invariants method outlined in sect. 3.3. A possible basis set, picked from the $V \otimes A \to V \otimes A$ linearly independent tree invariants, consists of

 (9.62)

The multiplication table (3.39) has been worked out in (9.54), (9.56) and (9.57). For example, the $(t_\alpha)_\beta{}^\gamma$ matrix rep for $Q\mathbf{t}$ is

$$\sum_{\gamma \in \mathcal{T}} (Q)_\beta{}^\gamma \mathbf{t}_\gamma = Q\begin{pmatrix}\mathbf{e}\\ R\\ Q\end{pmatrix} = \begin{pmatrix}0 & 0 & 1\\ 0 & -1/n & 0\\ 1 & -1/n & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}\\ R\\ Q\end{pmatrix} \qquad (9.63)$$

and similarly for R. In this way, we obtain the $[3\times 3]$ matrix rep of the algebra of invariants

$$\{\mathbf{e}, R, Q\} = \left\{\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1 & 0\\ 0 & n-\frac{1}{n} & 0\\ 0 & -1/n & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 1\\ 0 & -1/n & 0\\ 1 & -1/n & 0\end{pmatrix}\right\}. \qquad (9.64)$$

From (9.54) we already know that the eigenvalues of R are $\{0, 0, n - \frac{1}{n}\}$. The last eigenvalue yields the projection operator $P_1 = n/(n^2-1)$, but the projection operator P_4 yields a 2-dimensional degenerate rep. Q has 3 distinct eigenvalues $\{-1/n, 1, -1\}$ and is thus more interesting; the corresponding projection operators fully decompose the $V \otimes A$ space. $-1/n$ eigenspace projection operator is again P_1, but P_4 is split into 2 subspaces, verifying (9.60) and (9.59):

$$P_2 = \frac{(Q+\mathbf{1})(Q+\frac{1}{n}\mathbf{1})}{(1+1)(1+\frac{1}{n})} = \frac{1}{2}\left(\mathbf{1} + Q - \frac{1}{n+1}R\right)$$

$$P_3 = \frac{(Q-\mathbf{1})(Q+\frac{1}{n}\mathbf{1})}{(-1-1)(-1+\frac{1}{n})} = \frac{1}{2}\left(\mathbf{1} - Q - \frac{1}{n-1}R\right) . \qquad (9.65)$$

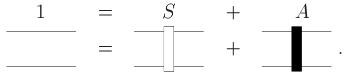
We see that the matrix rep of the algebra of invariants is an alternative tool for implementing the full reduction, perhaps easier to implement as computation than out and out birdtracks manipulations.

To summarize, the invariant matrix R projects out the 1-particle subspace P_1. The particle exchange matrix Q splits the remainder into the irreducible particle-adjoint subspaces P_2 and P_3.

9.10 TWO-INDEX ADJOINT TENSORS

Consider the Kronecker product of two adjoint reps. We want to reduce the space of tensors $x_{ij} \in A \otimes A$, with $i = 1, 2, \ldots N$. The first decomposition is the obvious decomposition (9.4) into the symmetric and antisymmetric subspaces

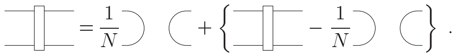
$$1 = S + A$$



(9.66)

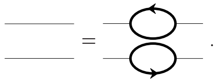
The symmetric part can be split into the trace and the traceless part, as in (9.45):

$$S = \frac{1}{N}T + P_S$$



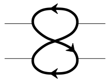
(9.67)

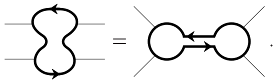
Further decomposition can be effected by studying invariant matrices in the $V^2 \otimes \overline{V}^2$ space. We can visualize the relation < between $A \otimes A$ and $V^2 \otimes \overline{V}^2$ by the identity



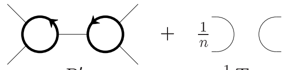
(9.68)

This suggests introduction of two invariant matrices

$Q =$  (9.69)

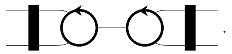
$R =$  (9.70)

R can be decomposed by (9.45) into a singlet and the adjoint rep

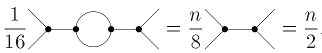
$R =$  (9.71)

$$= R' + \frac{1}{n}T .$$

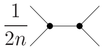
The singlet has already been taken into account in the trace-traceless tensor decomposition (9.67). R' projection on the antisymmetric subspace is

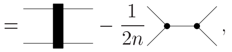
$AR'A =$  . (9.72)

By the Lie algebra (4.45)

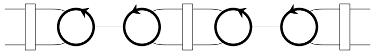
$(AR'A)^2 = \frac{1}{16}$  $= \frac{n}{2}AR'A$, (9.73)

and the associated projection operators

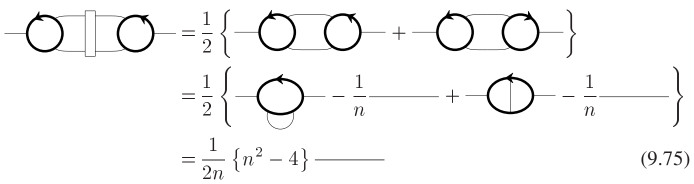
$(P_5)_{ij,kl} = \frac{1}{2n}C_{ijm}C_{mlk} = \frac{1}{2n}$ 

$P_a =$  , (9.74)

split the antisymmetric subspace into the adjoint rep and a reminder. On the symmetric subspace (9.67), R' acts as $P_S R' P_S$. As $R'T = 0$, this is the same as $SR'S$. Consider

$(SR'S)^2 =$  .

We compute

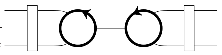


(9.75)

Hence, $SR'S$ satisfies the characteristic equation

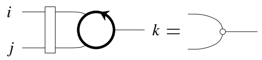
$$\left(SR'S - \frac{n^2-4}{2n}\right) SR'S = 0 . \qquad (9.76)$$

The associated projection operators split up the traceless symmetric subspace (9.67) into the adjoint rep and a reminder

$P_2 = \frac{2n}{n^2-4}SR'S = \frac{2n}{n^2-4}$  , (9.77)

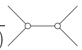
$$P_{2'} = P_S - P_2 . \qquad (9.78)$$

The Clebsch-Gordan coefficients for P_2 are known as the Gell-Mann d_{ijk} tensors [73]

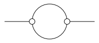
 $= \frac{1}{2}d_{ijk}$. (9.79)

(For $SU(3)$, P_2 is the projection operator ($\underline{8} \otimes \underline{8}$) symmetric $\to \underline{8}$). In terms of d_{ijk}'s, we have

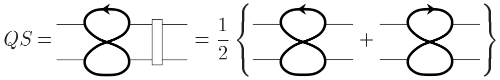
$$(P_2)_{ij,k\ell} = \frac{n}{2(n^2-4)} d_{ijm} d_{mk\ell}$$

$$= \frac{n}{2(n^2-4)}$$  , (9.80)

with the normalization

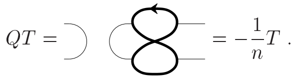
$$d_{ijk} d_{kj\ell} =$$  $$= \frac{2(n^2-4)}{n} \delta_{i\ell} \, . \qquad (9.81)$$

Next we turn to the decomposition of the symmetric subspace induced by matrix Q (9.69). Q commutes with S



$$= SQ = SQS \, . \qquad (9.82)$$

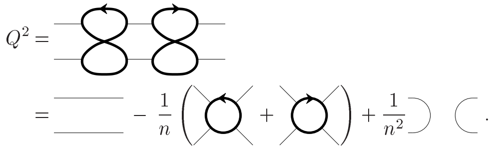
On the 1-dimensional subspace in (9.67), it takes eigenvalue $-1/n$

 (9.83)

So Q also commutes with the projection operator P_S from (9.67)

$$QP_S = Q\left(S - \frac{1}{n^2-1} T\right) = P_S Q \, . \qquad (9.84)$$

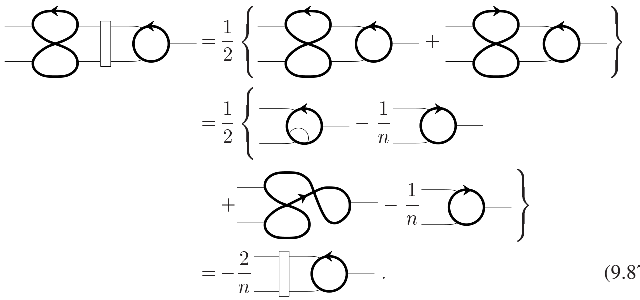
Q^2 is easily evaluated by inserting the adjoint rep projection operators (9.45)

 (9.85)

Projecting on the traceless symmetric subspace gives

$$P_S\left(Q^2 - 1 + \frac{n^2-4}{n^2} P_1\right) = 0 \, . \qquad (9.86)$$

On P_2 subspace Q gives

 (9.87)

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Hence, Q has a single eigenvalue

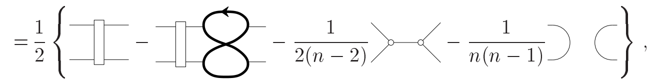
$$QP_2 = -\frac{2}{n} P_2 \qquad (9.88)$$

and does not decompose the P_2 subspace; this is as it should be, as P_2 is the adjoint rep and is thus irreducible. On $P_{2'}$ subspace (9.85) yields a characteristic equation

$$P_{2'}(Q^2 - 1) = 0 \, ,$$

with the associated projection operators

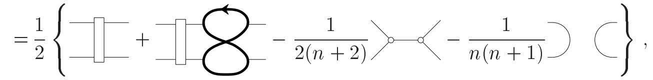
$$P_3 = \frac{1}{2} P_{2'}(1 - Q) \qquad (9.89)$$



$$P_4 = \frac{1}{2} P_{2'}(1 + Q) = \frac{1}{2}(P_S - P_1)(1 + Q)$$

$$= \frac{1}{2}\left(P_S - P_1 + SQ - \frac{1}{n^2-1} TQ + \frac{2}{n} P_1\right)$$

$$= \frac{1}{2}\left(S + SQ - \frac{n-2}{n} P_1 - \frac{1}{n(n+1)} T\right) \qquad (9.90)$$



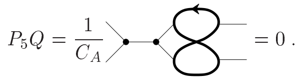
and dimensions tabulated in table 9.6. This completes the reduction of the symmetric subspace in (9.66). As in (9.82), Q commutes with A

$$QA = AQ = AQA \, . \qquad (9.91)$$

On the antisymmetric subspace, the Q^2 equation (9.85) becomes

$$0 = A(Q^2 - 1 + \frac{2}{n} R) \qquad A = A(Q^2 - 1 - P_A) \, . \qquad (9.92)$$

The adjoint rep (9.74) should be irreducible. Indeed, it follows from the Lie algebra, that Q has zero eigenvalue for any simple group:

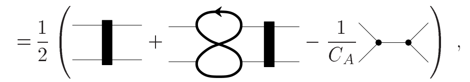
 (9.93)

On the remaining antisymmetric subspace P_a (9.92) yields the characteristic equation

$$P_a(Q^2 - 1) = 0 \, , \qquad (9.94)$$

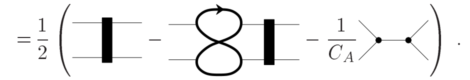
with corresponding projection operators

$$P_6 = \frac{1}{2} P_a(1 + Q) = \frac{1}{2} A(a + Q - P_A)$$



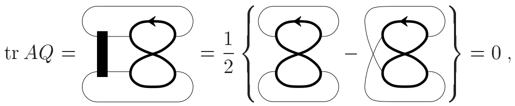
(9.95)

$$P_7 = \frac{1}{2} P_a (1 - Q)$$



(9.96)

To compute the dimensions of these reps we need

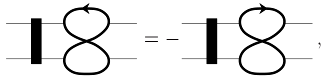


(9.97)

so both reps have the same dimension

$$d_6 = d_7 = \frac{1}{2}(\mathrm{tr}\, A - \mathrm{tr}\, P_A) = \frac{1}{2}\left\{\frac{(n^2-1)(n^2-2)}{2} - n^2 - 1\right\}$$
$$= \frac{(n^2-1)(n^2-4)}{4}\,. \qquad (9.98)$$

Indeed, the two reps are conjugate reps. The identity



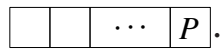
(9.99)

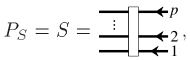
obtained by interchanging the two left adjoint rep legs, implies that the projection operators (9.95) and (9.96) are related by the reversal of the loop arrow. This is the birdtrack notation for complex conjugation.

This decomposition of two $SU(n)$ adjoint reps is summarized in table 9.6 and table 9.7.

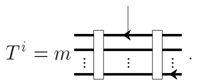
9.11 CASIMIRS FOR THE FULLY SYMMETRIC REPS OF $SU(N)$

In this section we carry out a few explicit birdtrack casimir evaluations.

Consider the fully symmetric Kronecker product of p particle reps. Its Dynkin label (defined on page 89) is $(p, 0, 0 \ldots 0)$, and the corresponding Young tableau is a row of p boxes: . The projection operator is given by (6.4)

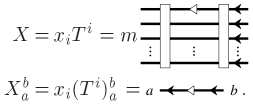


and the generator (4.39) in the symmetric rep is



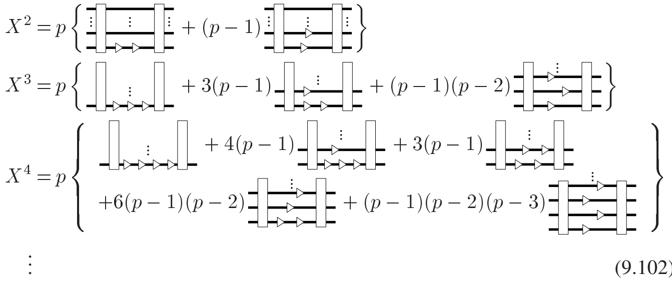
(9.100)

To compute the casimirs, we introduce matrices



(9.101)

We next compute the powers of X:



(9.102)

The $\mathrm{tr}\, X^k$ are then

$$\mathrm{tr}\, X^0 = d_s \binom{n+p-1}{p} \qquad \text{- see (6.13)} \qquad (9.103)$$

$$\mathrm{tr}\, X = 0 \qquad \text{(semi-simplicity)} \qquad (9.104)$$

$$\mathrm{tr}\, X^2 = d_s \frac{p(p+n)}{n(n+1)} \mathrm{tr}\, x^2 \qquad (9.105)$$

$$\mathrm{tr}\, X^3 = \frac{d_s}{n} p \left(1 + 3\frac{p-1}{n+1} + 2\frac{(p-1)(p-2)}{(n+1)(n+2)}\right) \mathrm{tr}\, x^3$$
$$= \frac{(n+p)!(n+2p)}{(n+2)!(p-1)!} \mathrm{tr}\, x^3 = d_s \frac{p(n+p)(n+2p)}{n(n+1)(n+2)} \mathrm{tr}\, x^3 \qquad (9.106)$$

$$\mathrm{tr}\, X^4 = d\frac{p}{n}\left(\left(1 + 7\frac{p-1}{n+1} + 12\frac{p-1}{n+1}\frac{p-2}{n+2} + 6\frac{p-1}{n+1}\frac{p-2}{n+2}\frac{p-3}{n+3}\right)\mathrm{tr}\, x^4\right.$$
$$\left. + \frac{p-1}{n+1}\left(3 + 6\frac{p-2}{n+2} + 3\frac{p-2}{n+2}\frac{p-3}{n+3}\right)\left(\mathrm{tr}\, x^2\right)^2\right)\,. \qquad (9.107)$$

The quadratic Dynkin index is given by the ratio of $\mathrm{tr}\, X^2$ and $\mathrm{tr}_A X^2$ for the adjoint rep (7.29):

$$\ell_2 = \frac{\mathrm{tr}\, X^2}{\mathrm{tr}_A X^2} = \frac{d_s p(p+n)}{2n^2(n+1)}\,. \qquad (9.108)$$

To take a random example from Patera-Sankoff tables [128]; the $SU(6)$ rep dimension and Dynkin index

rep	dim	ℓ_2
(0,0,0,0,0,14)	11628	6460 ,

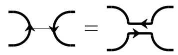
(9.109)

check with the above expressions.

9.12 $SU(N), U(N)$ EQUIVALENCE IN ADJOINT REP

The following simple observation speeds up evaluation of pure adjoint rep group-theoretic weights ($3n$-j)'s for $SU(n)$: The adjoint rep weights for $U(n)$ and $SU(n)$ are identical. This means that we can use the $U(n)$ adjoint projection operator

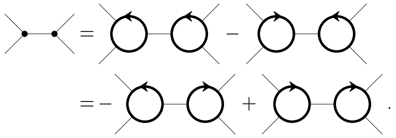
$U(n)$:



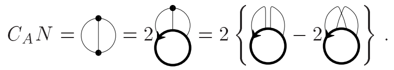
(9.110)

instead of the traceless $SU(n)$ projection operator (9.45), and halve the number of terms in the expansion of each adjoint line.

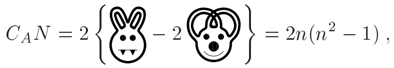
Proof: any internal adjoint line connects two C_{ijk}'s:



The trace part of (9.45) cancels on each line, hence, it does not contribute to the pure adjoint rep diagrams. As an example, we re-evaluate the adjoint quadratic casimir for $SU(n)$:



Now substitute the $U(n)$ adjoint projection operator (9.110):



in agreement with the first exercise of sect. 2.2.

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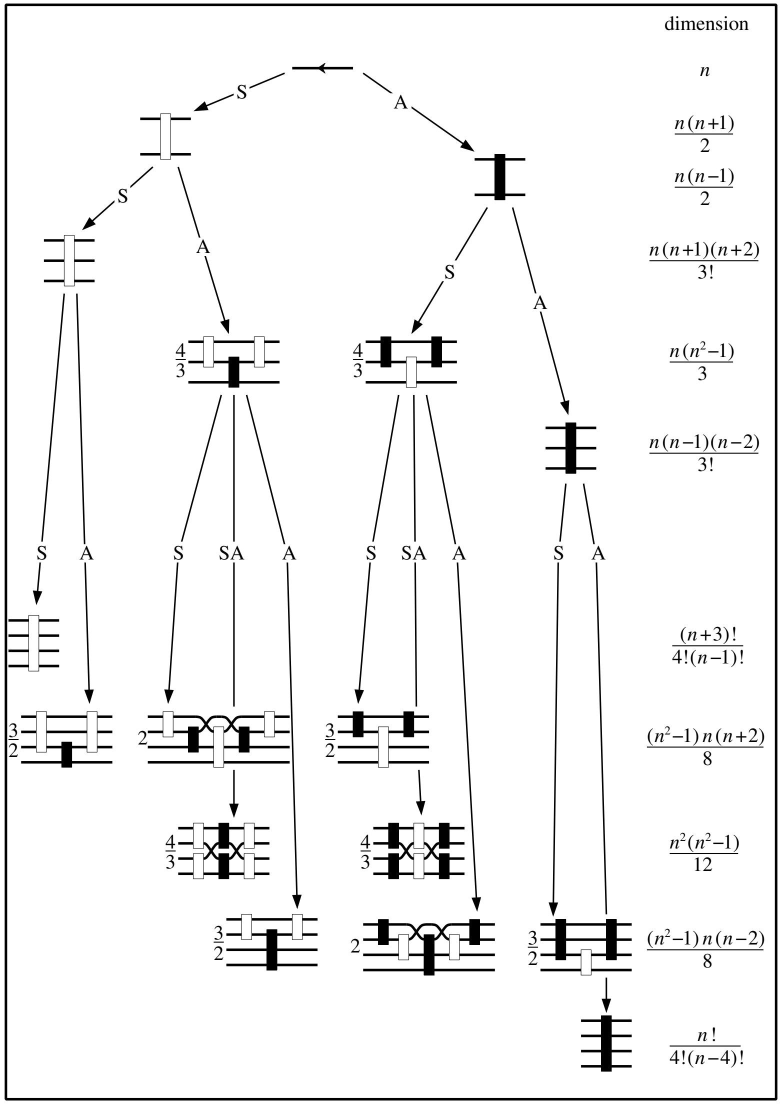

Table 9.1 Projection operators for 2-, 3- and 4-index tensors in $U(n)$, $SU(n)$, $n \geq p$ (p = number of indices)

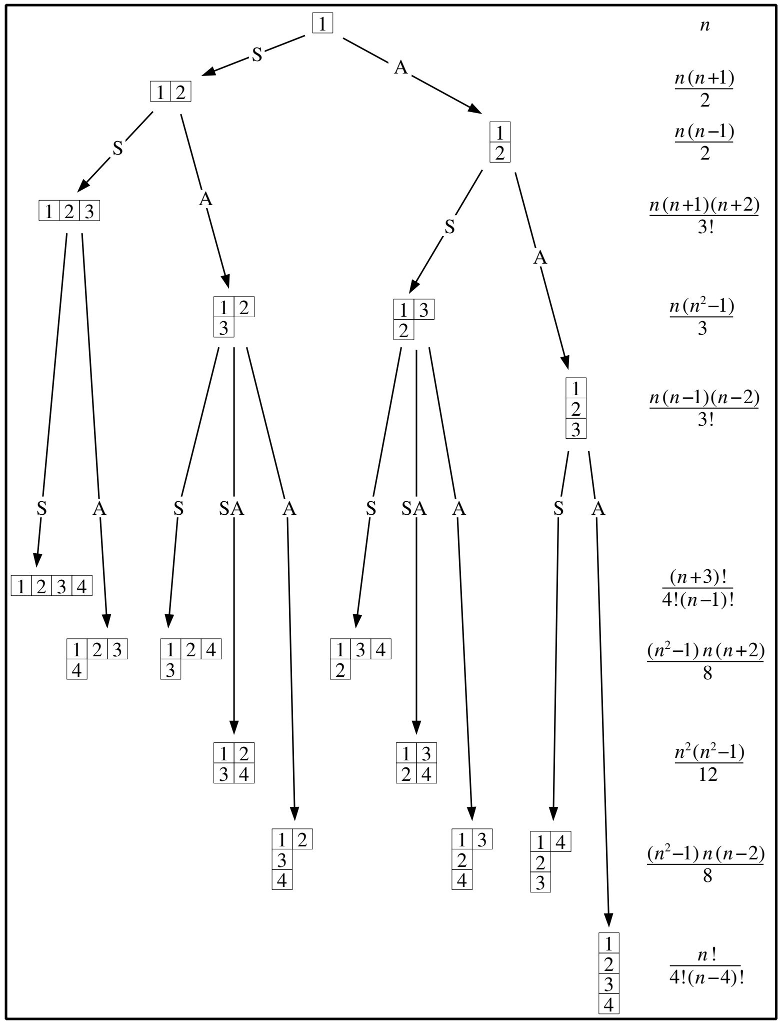

Table 9.2 Young tableaux for the irreducible reps of the symmetric group for 2-, 3- and 4-index tensors. Rows correspond to symmetrizations, columns to antisymmetrizations. The reduction procedure is not unique, as it depends on the order in which the indices are combined; this order is indicated by labels 1, 2, 3 , ..., p in the boxes of Young tableaux.

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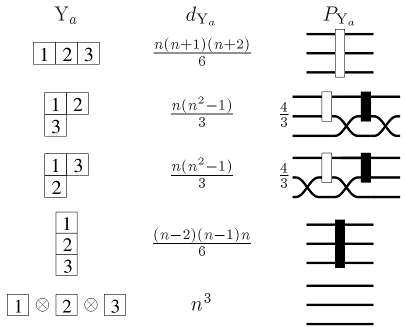


Table 9.3 Reduction of 3-index tensor. The bottom row is the direct sum of the Young tableaux, the sum of the dimensions, and the sum of the projection operators, verifying the completeness (3.48).

Y_a	d_{Y_a}	P_{Y_a}
$\begin{smallmatrix}1&2&3&4\end{smallmatrix}$	$\frac{n(n+1)(n+2)(n+3)}{24}$	
$\begin{smallmatrix}1&2&3\\4\end{smallmatrix}$	$\frac{(n-1)n(n+1)(n+2)}{8}$	$\frac{3}{2}$
$\begin{smallmatrix}1&2&4\\3\end{smallmatrix}$	$\frac{(n-1)n(n+1)(n+2)}{8}$	$\frac{3}{2}$
$\begin{smallmatrix}1&3&4\\2\end{smallmatrix}$	$\frac{(n-1)n(n+1)(n+2)}{8}$	$\frac{3}{2}$
$\begin{smallmatrix}1&2\\3&4\end{smallmatrix}$	$\frac{n^2(n^2-1)}{12}$	$\frac{4}{3}$
$\begin{smallmatrix}1&3\\2&4\end{smallmatrix}$	$\frac{n^2(n^2-1)}{12}$	$\frac{4}{3}$
$\begin{smallmatrix}1&2\\3\\4\end{smallmatrix}$	$\frac{(n-1)n(n+1)(n+2)}{8}$	$\frac{3}{2}$
$\begin{smallmatrix}1&3\\2\\4\end{smallmatrix}$	$\frac{(n-2)(n-1)n(n+1)}{8}$	$\frac{3}{2}$
$\begin{smallmatrix}1&4\\2\\3\end{smallmatrix}$	$\frac{(n-2)(n-1)n(n+1)}{8}$	$\frac{3}{2}$
$\begin{smallmatrix}1\\2\\3\\4\end{smallmatrix}$	$\frac{(n-2)(n-1)n(n+1)}{8}$	
$\boxed{1}\otimes\boxed{2}\otimes\boxed{3}\otimes\boxed{4}$	n^4	

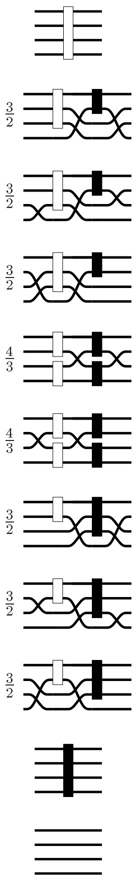
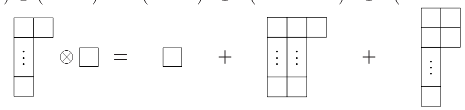


Table 9.4 Reduction of 4-index tensors.

	$\underline{A}\otimes\underline{q}$	=	$\underline{\lambda_1}$	$\oplus$	$\underline{\lambda_2}$	$\oplus$	$\underline{\lambda_3}$
Dynkin labels:	$(10\dots01)\otimes(10\dots)$	=	$(10\dots)$	$\oplus$	$(100\dots02)$	$\oplus$	$(010\dots01)$
Dimensions:	$(n^2-1)n$	=	n	+	$\frac{n(n-1)(n-2)}{2}$	+	$\frac{n(n+1)(n-2)}{2}$
Indices:	$n+\frac{n^2-1}{2n}$	=	$\frac{1}{2n}$	+	$\frac{(n+2)(3n-1)}{4n}$	+	$\frac{(n-2)(3n+1)}{4n}$



SU(3):

Dimensions:	$8\cdot 3$	=	3	+	15	+	6
Indices:	$13/3$	=	$1/6$	+	$10/3$	+	$5/6$

SU(4):

Dimensions:	$15\cdot 4$	=	4	+	36	+	20
Indices:	$47/8$	=	$1/8$	+	$33/8$	+	$13/8$

Projection operators:

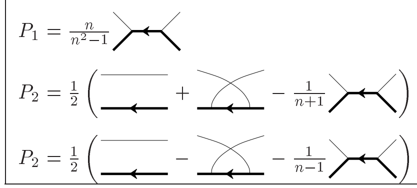


Table 9.5 $SU(n)$ $V\otimes A$ Clebsch-Gordan series.

			Symmetric								Antisymmetric				
	$\underline{\lambda}_A \otimes \underline{\lambda}_A$	$=$	λ_1	$\oplus$	λ_2	$\oplus$	λ_3	$\oplus$	λ_4	$\oplus$	λ_5	$\oplus$	λ_6	$\oplus$	λ_7
Dimensions	$(n^2-1)^2$	$=$	1	$+$	(n^2-1)	$+$	$\frac{n^2(n-3)(n+1)}{4}$	$+$	$\frac{n^2(n+3)(n-1)}{4}$	$+$	(n^2-1)	$+$	$\frac{(n^2-1)(n^2-4)}{4}$	$+$	$\frac{(n^2-1)(n^2-4)}{4}$
Dynkin indices	$2(n^2-1)$	$=$	0	$+$	1	$+$	$\frac{n(n-3)}{2}$	$+$	$\frac{n(n+3)}{2}$	$+$	1	$+$	$\frac{n^2-4}{2}$	$+$	$\frac{n^2-4}{2}$
SU(3) example:															
Dimensions	8^2	$=$	1	$+$	8	$+$	0	$+$	27	$+$	8	$+$	10	$+$	$\overline{10}$
Indices	$2\cdot 8$	$=$	0	$+$	1	$+$	0	$+$	9	$+$	1	$+$	$\frac{5}{2}$	$+$	$\frac{5}{2}$
SU(4) example:															
	$(101)\otimes(101)$	$=$	(000)	$\oplus$	(101)	$\oplus$	(020)	$\oplus$	(202)	$\oplus$	(101)	$\oplus$	(012)	$\oplus$	(210)
Dimensions	15^2	$=$	1	$+$	15	$+$	20	$+$	84	$+$	15	$+$	45	$+$	$\overline{45}$
Indices	$2\cdot 15$	$=$	0	$+$	1	$+$	2	$+$	14	$+$	1	$+$	6	$+$	6

Projection operators

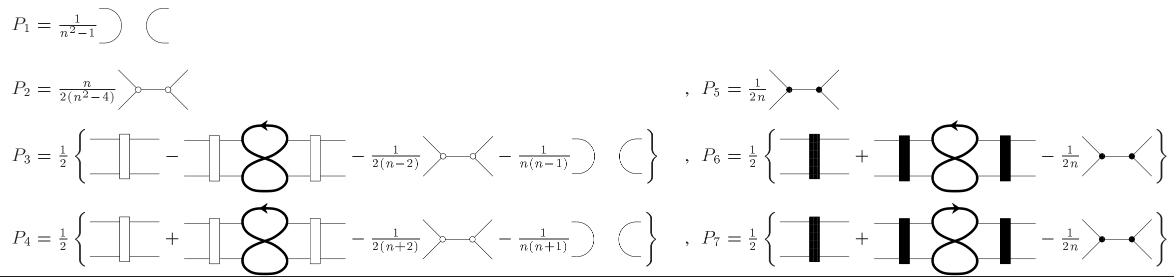


Table 9.7 SU(n), $n \geq 3$ Clebsch-Gordan series for A $\otimes$ A.

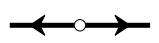
AJ: create table 8.5 from manuscript

Table 9.6 Summary of the reduction of a Kronecker product of two $SU(n)$ adjoint reps. The flip matrix F induces decomposition into symmetric and antisymmetric subspaces (9.66). The trace matrix T projects out the singlet rep (9.67). R' from (9.70) projects the adjoint reps in both the symmetric and antisymmetric subspaces. Finally, the interchange matrix Q from (9.69) decomposes the $P_{2'}$ and P_a subspaces.

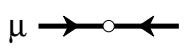
Chapter Ten

Orthogonal groups

Orthogonal group $SO(n)$ is the group of transformations which leave invariant a symmetric quadratic form $(q,q) = g_{\mu\nu} q^\mu q^\nu$:

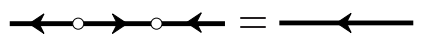
$$g_{\mu\nu} = g_{\nu\mu} = \mu$$  $$\nu \qquad \mu, \nu = 1, 2, \ldots, n\,. \tag{10.1}$$

If (q,q) is an invariant, so is its complex conjugate $(q,q)^* = g^{\mu\nu} q_\mu q_\nu$, and

$$g^{\mu\nu} = g^{\nu\mu} = \mu$$  $$\nu \tag{10.2}$$

is also an invariant tensor. Matrix $A_\mu^\nu = g_{\mu\sigma} g^{\sigma\nu}$ must be proportional to unity, as otherwise its characteristic equation would decompose the defining n-dimensional rep. A convenient normalization is

$$g_{\mu\sigma} g^{\sigma\nu} = \delta_\mu^\nu$$

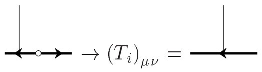
 $$. \tag{10.3}$$

As the indices can be raised and lowered at will, nothing is gained by keeping the arrows. Our convention will be to perform all contractions with metric tensors with upper indices and omit the arrows and the open dots:

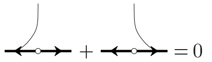
$$g^{\mu\nu} \equiv \mu \;\rule{2em}{0.4pt}\; \nu\,. \tag{10.4}$$

All other tensors will have lower indices. For example, Lie group generators $(T_i)_\mu{}^\nu$ from (4.29) will be replaced by

$$(T_i)_\mu{}^\nu = \quad \to (T_i)_{\mu\nu} = \quad .$$

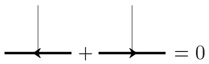


The invariance condition (4.35) for the metric tensor



$$(T_i)_\mu{}^\sigma g_{\sigma\nu} + (T_i)_\nu{}^\sigma g_{\mu\sigma} = 0 \tag{10.5}$$

becomes, in this convention, a statement that the $SO(n)$ generators are antisymmetric:



$$(T_i)_{\mu\nu} = -(T_i)_{\nu\mu}\,. \tag{10.6}$$

Our analysis of the reps of $SO(n)$ will depend only on the existence of a symmetric metric tensor and its invertability, and not on its eigenvalues. The resulting Clebsch-Gordan series applies both to the compact $SO(n)$ and non-compact orthogonal groups, such as the Minkowski group $SO(1,3)$. In this chapter, we outline the construction of $SO(n)$ tensor reps. Spinor reps will be taken up in chapter 11.

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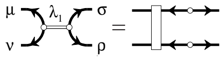
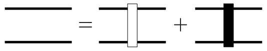
10.1 TWO-INDEX TENSORS

In sect. 9.1 we have decomposed the $SU(n)$ 2-index tensors into symmetric and antisymmetric parts. For $SO(n)$, the rule is to lower all indices on all tensors, and the symmetric state projection operator (9.2) is replaced by

$$S_{\mu\nu,\rho\sigma} = g_{\rho\rho'} g_{\sigma\sigma'} S_{\mu\nu,}{}^{\rho'\sigma'}$$
$$= \frac{1}{2}\left(g_{\mu\sigma}g_{\nu\rho} + g_{\mu\rho}g_{\nu\sigma}\right)$$



From now on, we drop all arrows and $g^{\mu\nu}$'s and write (9.4) as

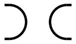


$$g_{\mu\sigma}g_{\nu\rho} = \frac{1}{2}(g_{\mu\sigma}g_{\nu\rho} + g_{\mu\rho}g_{\nu\sigma}) + \frac{1}{2}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma})\,. \tag{10.7}$$

The new invariant, specific to $SO(n)$, is the index contraction:

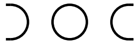
$$T_{\mu\nu,\rho\sigma} = g_{\mu\nu} g_{\rho\sigma}$$

$$T = \quad . \tag{10.8}$$



This invariant satisfies a trivial characteristic equation

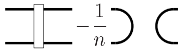
$$T^2 = \quad = nT\,, \tag{10.9}$$



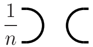
which yields the trace and the traceless part projection operators (9.44), (9.45). As T is symmetric, $ST = T$, only the symmetric subspace is resolved by this invariant. The final decomposition of $SO(n)$ 2-index tensors is

traceless symmetric:

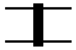
$$(P_2)_{\mu\nu,\rho\sigma} = \frac{1}{2}\left(g_{\mu\sigma}g_{\nu\rho} + g_{\mu\rho}g_{\nu\sigma}\right) - \frac{1}{n} g_{\mu\nu}g_{\rho\sigma} = \quad - \frac{1}{n} \quad , \tag{10.10}$$



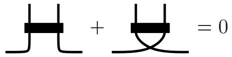
singlet: $$(P_1)_{\mu\nu,\rho\sigma} = \frac{1}{n} g_{\mu\nu}g_{\rho\sigma} = \frac{1}{n} \quad , \tag{10.11}$$



antisymmetric: $$(P_3)_{\mu\nu,\rho\sigma} = \frac{1}{2}\left(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}\right) = \quad . \tag{10.12}$$

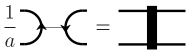


The adjoint rep (9.49) of $SU(n)$ is decomposed into the traceless symmetric and the antisymmetric parts. To determine which of them is the new adjoint rep, we substitute them into the invariance condition (10.5). Only the antisymmetric projection operator satisfies the invariance condition



so the adjoint rep projection operator for $SO(n)$ is

$$\frac{1}{a} \quad = \quad . \tag{10.13}$$



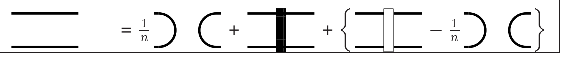
Young tableaux	$\square \times \square$	=	$\bullet$	+	(two-box column)	+	(two-box row)
Dynkin labels	$(10\ldots)\times(10\ldots)$	=	$(00\ldots)$	+	$(010\ldots)$	+	$(20\ldots)$
Dimensions	n^2	=	1	+	$\frac{n(n-1)}{2}$	+	$\frac{(n+2)(n-1)}{2}$
Dynkin indices	$2n\frac{1}{n-2}$	=	0	+	1	+	$\frac{n+2}{n-2}$
Projectors							

Table 10.1 $SO(n)$ Clebsch-Gordan series for $V \otimes V$.

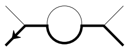
The dimension of $SO(n)$ is given by the trace of the adjoint projection operator:

$$N = \mathrm{tr}\, P_A =$$  $$= \frac{n(n-1)}{2} . \qquad (10.14)$$

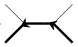
Dimensions of the other reps and the Dynkin indices (see sect. 7.4) are listed in table 10.1.

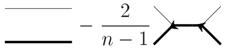
10.2 MIXED ADJOINT ⊗ DEFINING REP TENSORS

The mixed adjoint-defining rep tensors are decomposed in the same way as for $SU(n)$. The intermediate defining rep state matrix R (9.52) satisfies the characteristic equation

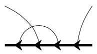
$$R^2 =$$  $$= \frac{n-1}{2} R . \qquad (10.15)$$

The corresponding projection operators are

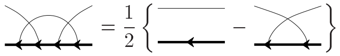
$$P_1 = \frac{2}{n-1}$$  ,

$$P_2 = \;\; - \frac{2}{n-1}$$  . (10.16)

The eigenvalue of Q from (9.53) on the defining subspace can be computed by inserting the adjoint projection operator (10.13):

$$QR =$$  $$= \frac{1}{2} R . \qquad (10.17)$$

Q^2 is also computed by inserting (10.13):

$$Q^2 =$$  $$= \frac{1}{2}(1-Q) . \qquad (10.18)$$

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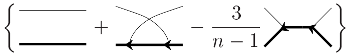
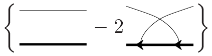
The eigenvalues are $\{-1, \frac{1}{2}\}$, and the associated projection operators (3.45) are

$$P_2 = P_4 \frac{2}{3}(1+Q) = \frac{2}{3}\left(1 - \frac{2}{n-1}R\right)(1+Q) = \frac{2}{3}\left(1 + Q - \frac{3}{n-1}R\right)$$

$$= \frac{2}{3}\left\{ \;\; + \;\; - \frac{3}{n-1} \;\; \right\} , \qquad (10.19)$$



$$P_3 = P_4 \frac{1}{3}(1-2Q) = \frac{1}{3}\left\{ \;\; - 2 \;\; \right\} . \qquad (10.20)$$

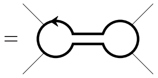


This decomposition is summarized in table 10.2.

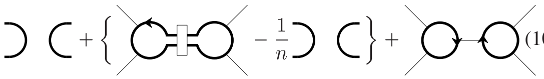
The same decomposition can be obtained by viewing the $SO(n)$ defining-adjoint tensors as  $\times \square$ products, and starting with the $SU(n)$ decomposition along the lines of sect. 9.2.

10.3 TWO-INDEX ADJOINT TENSORS

The reduction of the 2-index adjoint rep tensors proceeds as for $SU(n)$. The annihilation matrix R (9.70) induces decomposition (10.11)-(10.12) into three tensor spaces

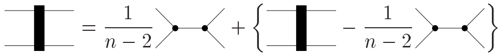
$$R =$$ 

$$= \frac{1}{n} \;\; + \left\{ \;\; - \frac{1}{n} \;\; \right\} + \qquad (10.21)$$



On the antisymmetric subspace, the last term projects out the adjoint rep:

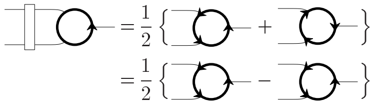
$$= \frac{1}{n-2} \;\; + \left\{ \;\; - \frac{1}{n-2} \;\; \right\} . \qquad (10.22)$$



The last term in (10.21) does not affect the symmetric subspace

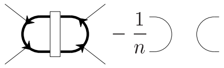
$$= \frac{1}{2}\left\{ \;\; + \;\; \right\}$$

$$= \frac{1}{2}\left\{ \;\; - \;\; \right\} = 0 \qquad (10.23)$$



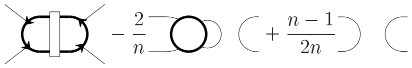
because of the antisymmetry of the $SO(n)$ generators ($d_{ijk} = 0$ for orthogonal groups). The second term in (10.21)

$$R_S = \;\; - \frac{1}{n} \qquad (10.24)$$



projects out the intermediate symmetric 2-index tensors subspace. To normalize it, we compute R_S^2:

$$(R_S)^2 = \;\; - \frac{2}{n} \;\; + \frac{n-1}{2n}$$



$$= \frac{n-2}{4} R_S . \qquad (10.25)$$

Row							
Dynkin labels	$(010\ldots)\times(100\ldots)$	=	$(100\ldots)$	+	$(0010\ldots)$	+	$(110\ldots)$
Dimensions	$\frac{n^2(n-1)}{2}$	=	n	+	$\frac{n(n-1)(n-2)}{6}$	+	$\frac{n(n^2-4)}{3}$
$SO(3)$	$\underline{9}$	=	$\underline{3}$	+	$\underline{1}$	+	$\underline{5}$
$SO(4)$	$\underline{24}$	=	$\underline{4}$	+	$\underline{4}$	+	$\underline{16}$
Dynkin indices	?	=	?		?		?

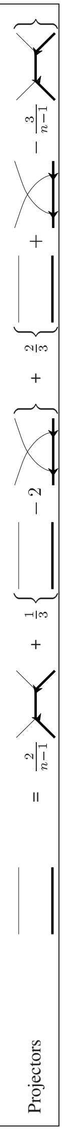


Table 10.2 $SO(n)$ $A \otimes V$ Clebsch-Gordan series.

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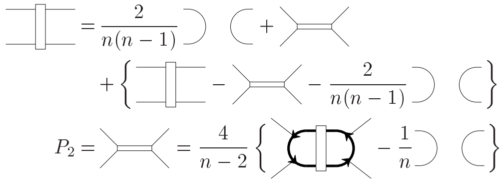
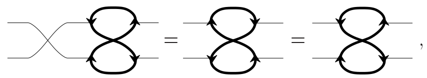
R_S decomposes the symmetric 2-index adjoint subspace into



(10.26)

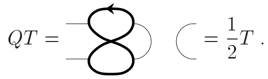
Because of the antisymmetry of the $SO(n)$ generators, the index interchange matrix (9.69) is symmetric

$$SQ = SQ^* = Q$$



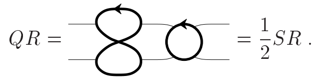
(10.27)

so it cannot induce a decomposition of the antisymmetric subspace in (10.22). Here Q^* indicates the diagram for Q with the arrow reversed. On the singlet subspace it has eigenvalue $\frac{1}{2}$:



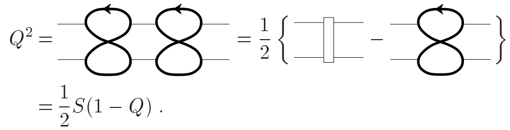
(10.28)

On the symmetric 2-index defining rep tensors subspace, its eigenvalue is also $\frac{1}{2}$, as the evaluation by the substitution of adjoint projection operators by (10.13) yields



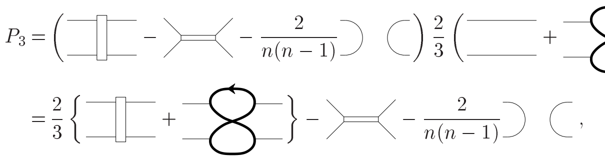
(10.29)

Q^2 is evaluated in the same manner:

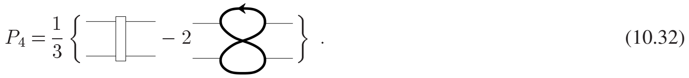


(10.30)

Thus, Q satisfies the same characteristic equation as in (10.18). The corresponding projection operators decompose the symmetric subspace (the third term in (10.26)) into



(10.31)



	A ⊗ A	=	λ_1	⊕	λ_2	⊕	λ_3	⊕	λ_4	⊕	λ_5	⊕	λ_6
			Symmetric								Antisymmetric		
Young tableaux	×	=	•	+		+		+		+		+	
Dynkin labels	(010...)×(010...)	=	(00...)	+	(20...)	+	(02...)	+	(00010..)	+	(010...)	+	(1010...)
Dimensions	$\frac{n^2(n-1)^2}{4}$	=	1	+	$\frac{(n-1)(n+2)}{2}$	+	$\frac{(n-3)n(n+1)(n+2)}{12}$	+	$\frac{n(n-1)(n-2)(n-3)}{24}$	+	$\frac{n(n-1)}{2}$	+	$\frac{n(n+2)(n-1)(n-3)}{8}$
Dynkin indices		=	0	+		+		+		+		+	
SO(3)	9	=	1	+	5	+	0	+	0	+	3	+	0
SO(4)	36	=	1	+	9	+	10	+	1	+	6	+	9
SO(5)=Sp(4)	100	=	1	+	14	+	35	+	5	+	10	+	35
SO(6)=SU(4)	225	=	1	+	20	+	84	+	15	+	15	+	(45+45)
SO(7)	441	=	1	+	27	+	168	+	35	+	21	+	189
SO(8)	784	=	1	+	35	+	300	+	70	+	28	+	350
SO(9)	1296	=	1	+	44	+	495	+	126	+	36	+	594
SO(10)	2025	=	1	+	54	+	770	+	210	+	45	+	945

Projection operators

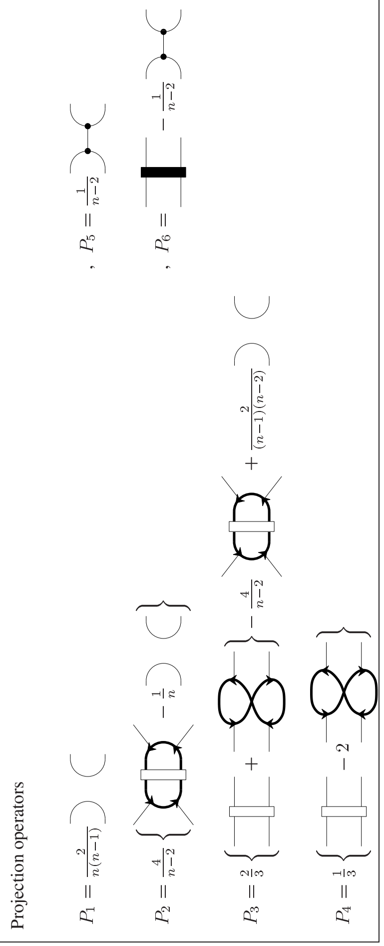


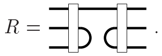
Table 10.3 $SO(n)$, $n \geq 3$ Clebsch-Gordan series for $A \otimes A$.

This Clebsch-Gordan series is summarized in table 10.3.

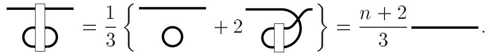
The reduction of 2-index adjoint tensors, outlined above, is patterned after the reduction for $SU(n)$. Another, fully equivalent approach, is to consider the $SO(n)$ 2-index adjoint tensors as × products and start from the decomposition of sect. 9.5.1. This will be partially carried out in sect. 10.5.

10.4 THREE-INDEX TENSORS

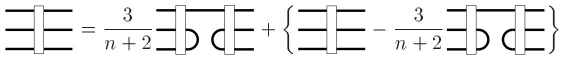
In the reduction of the 2-index tensors in sect. 10.1, the new $SO(n)$ invariant was the index contraction (10.8). In general, for a multi–index tensor, the $SU(n) \to SO(n)$ reduction is due to the additional index contraction invariants. Consider the fully symmetric 3-index $SU(n)$ state in table 9.3. The new $SO(n)$ invariant matrix on this space is

 (10.33)

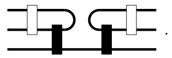
This is a projection onto the defining rep. The normalization follows from

 (10.34)

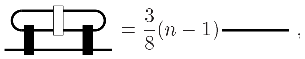
The rep of $SU(n)$ thus splits into

 (10.35)

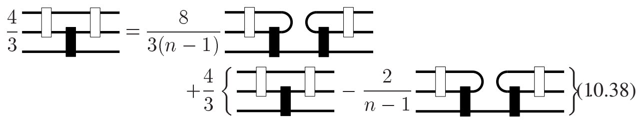
On the mixed symmetry subspace in table 9.3, one can try various index contraction matrices R_i. However, their projections $P_2 R_i P_2$ are all proportional to

 (10.36)

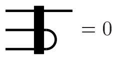
The normalization is fixed by

 (10.37)

and the mixed symmetry rep of $SU(n)$ in (9.12) splits as

 (10.38)

The other mixed symmetry rep in table 9.3 splits in analogous fashion. The fully antisymmetric space is not affected by contractions, as

 (10.39)

by the symmetry of $g_{\mu\nu}$. Besides, as is the adjoint rep, we have already performed the × □ decomposition in the preceding section. The full Clebsch-Gordan series for the $SO(n)$ 3-index tensors is given in table 10.4.

	$V\otimes V\otimes V$		λ_1	$\oplus$	λ_2	$\oplus$	λ_3	$\oplus$	λ_4	$\oplus$	λ_5	$\oplus$	λ_6	$\oplus$	λ_7
											$\underbrace{\quad\quad\quad\quad\quad\quad\quad\quad}_{A\otimes V}$				
Young tableaux	$\boxed{1}\times\boxed{2}\times\boxed{3}$	=	$\square\square\square$	+	$\square$	+	$\begin{smallmatrix}1&2\\3&\end{smallmatrix}$	+	$\square$	+	$\begin{smallmatrix}1&3\\2&\end{smallmatrix}$	+	$\square$	+	$\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$
Dynkin labels			$(30\ldots)$	+	$(10\ldots)$	+	$(110\ldots)$	+	$(10\ldots)$	+	$(110\ldots)$	+	$(10\ldots)$	+	$(0010\ldots)$
Dimensions	n^3	=	$\frac{(n-1)n(n+4)}{6}$	+	n	+	$\frac{n(n^2-4)}{3}$	+	n	+	$\frac{n(n^2-4)}{3}$	+	n	+	$\frac{n(n-1)(n-2)}{6}$
SO(3)	27	=	7	+	3	+	5	+	3	+	5	+	3	+	1
SO(4)	64	=	16	+	4	+	16	+	4	+	16	+	4	+	4
Dynkin indices		=		+		+		+		+		+		+	

Projection operators

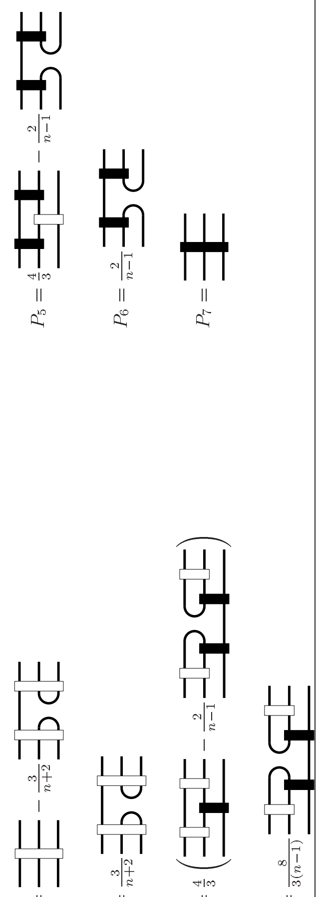


Table 10.4 $SO(n)$ Clebsch-Gordan series for $V\otimes V\otimes V$.

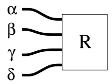
10.5 GRAVITY TENSORS

In a different application of birdtracks, we now change the language and construct the "irreducible rank-four gravity curvature tensors". The birdtrack notation for Young projection operators had originally been invented by Penrose [132] in this context. The Riemann-Christoffel curvature tensor has the following symmetries [155]:

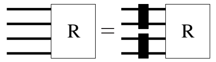
$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= -R_{\beta\alpha\gamma\delta} \\ R_{\alpha\beta\gamma\delta} &= R_{\gamma\delta\alpha\beta} \\ R_{\alpha\beta\gamma\delta} + R_{\beta\gamma\alpha\beta} &+ R_{\gamma\alpha\beta\delta} = 0\,. \end{aligned} \tag{10.40}$$

Introducing birdtrack notation for the Riemann tensor

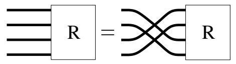
$$R_{\alpha\beta\gamma\delta} = \quad , \tag{10.41}$$



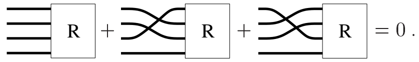
we can state the above symmetries as



(10.42)

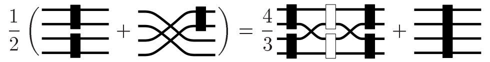


(10.43)



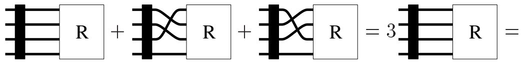
(10.44)

The first condition says that R lies in ⊗ subspace. We have decomposed this subspace in table 9.4. The second condition says that R lies in ↔ interchange-symmetric subspace, which splits into and subspaces:



(10.45)

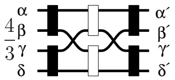
The third condition says that R has no components in the space:



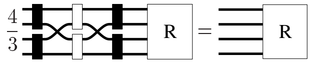
(10.46)

Hence, the Riemann tensor is a pure tensor, whose symmetries are summarized by the rep projection operator [132]:

$$(P_R)_{\alpha\beta\gamma\delta,}{}^{\delta'\gamma'\beta'\alpha'} = \frac{4}{3} \tag{10.47}$$



$$(P_R R)_{\alpha\beta\gamma\delta} = (P_R)_{\alpha\beta\gamma\delta,}{}^{\delta'\gamma'\beta'\alpha'} R_{\alpha'\beta'\gamma'\delta'} = R_{\alpha\beta\gamma\delta}$$



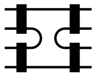
(10.48)

This compact statement of the Riemann tensor symmetries yields immediately the number of independent components of $R_{\alpha\beta\gamma\delta}$, *ie.* the dimension of the ⊞ reps in table 9.4:

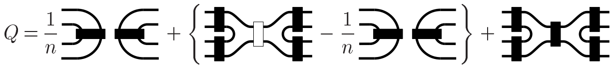
$$d_R = \operatorname{tr} P_R = \frac{n^2(n^2-1)}{12}\,. \tag{10.49}$$

The Riemann tensor has the symmetries of the ⊞ rep of $SU(n)$. However, gravity is also characterized by the symmetric tensor $g_{\alpha\beta}$ which induces local $SO(n)$ invariance (more precisely $SO(1, n-1)$, but compactness is not important here). The extra invariants built from $g_{\alpha\beta}$'s decompose $SU(n)$ reps into sums of $SO(n)$ reps.

The $SU(n)$ subspace, corresponding to ⊞, is decomposed by the $SO(n)$ intermediate 2-index state contraction matrix

$Q =$  . (10.50)

The intermediate 2-index subspace splits into three irreducible reps by (10.11)-(10.12)

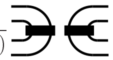


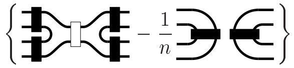
$= Q_0 + Q_S + Q_A\,.$ (10.51)

The Reimann tensor is symmetric under the interchange of index pairs, so the antisymmetric 2-index state does not contribute

$$P_R Q_A = 0\,. \tag{10.52}$$

The normalization of the remaining two projectors is fixed by computation of Q_S^2, Q_0^2:

$P_0 = \frac{2}{n(n-1)}$  , (10.53)

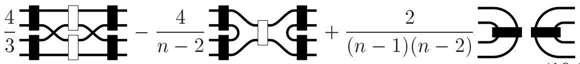
$P_S = \frac{4}{n-2}$  . (10.54)

This completes the $SO(n)$ reduction of the ⊞ $SU(n)$ rep (10.48):

	$SU(n)$	$\to$	$SO(n)$				
tableaux	⊞	$\to$	⊞	+	□□	+	∘
projectors	P_R	=	P_W	+	P_S	+	P_0
dimensions	$\frac{n^2(n^2-1)}{12}$	=	$\frac{(n+2)(n+1)n(n-3)}{12}$	+	$\frac{(n+2)(n-1)}{2}$	+	1

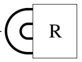
(10.55)

Here the projector for the traceless ⊞ tensor is given by $P_W = P_R - P_S - P_0$:

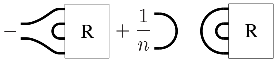
$P_W =$  . (10.56)

The above three projectors project out the standard relativity tensors:

Curvature scalar:

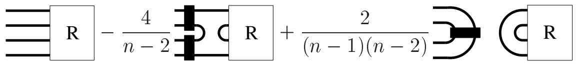
$R = -$  $= R^{\mu}{}_{\nu\mu}{}^{\nu}$ (10.57)

Traceless Ricci tensor:

$R_{\mu\nu} - \frac{1}{n} g_{\mu\nu} R = -$  (10.58)

Weyl tensor:

$C_{\lambda\mu\nu\kappa} = (P_W R)_{\lambda\mu\nu\kappa}$

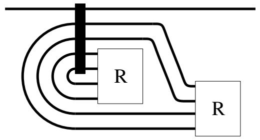
$=$ 

$$= R_{\lambda\mu\nu\kappa} + \frac{1}{n-2}\left(g_{\mu\nu}R_{\lambda\kappa} - g_{\lambda\nu}R_{\mu\kappa} - g_{\mu\kappa}R_{\lambda\nu} + g_{\lambda\kappa}R_{\mu\nu}\right) - \frac{1}{(n-1)(n-2)}(g_{\lambda\kappa}g_{\mu\nu} - g_{\lambda\nu}g_{\mu\kappa})R\,. \tag{10.59}$$

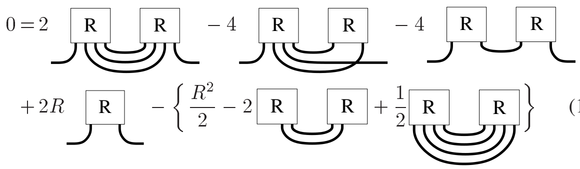
The numbers of independent components of these tensors are given by the dimensions of corresponding subspaces in (10.55). The Ricci tensor contributes first in three dimensions, and the Weyl tensor first in four, so we have

$$\begin{aligned} n=2: \quad R_{\lambda\mu\nu\kappa} &= (P_0R)_{\lambda\mu\nu\kappa} = \tfrac{1}{2}(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu})R \\ n=3: \quad &= g_{\lambda\nu}R_{\mu\kappa} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu} - g_{\lambda\kappa}R_{\mu\nu} \\ &\quad -\tfrac{1}{2}(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu})R\,. \end{aligned} \tag{10.60}$$

The last example of this section is an application of birdtracks to general relativity index manipulations. The object is to find the characteristic equation for the Riemann tensor in *four dimensions*. We contract (6.24) with two Riemann tensors

$0 =$  (10.61)

Expanding with (6.19) we obtain the characteristic equation

 (10.62)

For example, this identity has been used by Adler *et al.*, eq. (E2) in ref. [1].

10.6 $SO(N)$ DYNKIN LABELS

In general, one has to distinguish between the odd and the even dimensional orthogonal groups, as well as their spinor and non-spinor reps. In this chapter, we study only the tensor reps; spinor reps will be taken up in chapter 11.

For $SO(2r+1)$ reps there are r Dynkin labels $(a_1 a_2 \ldots a_{r-1} Z)$. If Z is odd, the rep is spinor; if Z is even, it is tensor. For the tensor reps, the corresponding Young tableau in the Fischler (B.11) notation is given by

$$(a_1 a_2 \ldots a_{r-1} Z) \to (a_1 a_2 \ldots a_{r-1} \frac{Z}{2} 00 \ldots) \,. \tag{10.63}$$

For example, for $SO(7)$ rep (102) we have

$$(102) \to (1010\ldots) = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \,. \tag{10.64}$$

For orthogonal groups, the Levi-Civita tensor can be used to convert a long column of k boxes into a short column of $(2r+1-k)$ boxes. The highest column which cannot be shortened by this procedure has r boxes, where r is the rank of $SO(2r+1)$.

For $SO(2r)$ reps, the last two Dynkin labels are spinor roots $(a_1 a_2 \ldots a_{r-2} Y Z)$. Tensor reps have $Y + Z$ =even. However, as spinors are complex, tensor reps can also be complex, conjugate reps being related by

$$(a_1 a_2 \ldots Y Z) = (a_1 a_2 \ldots Z Y)^* \,. \tag{10.65}$$

For $Z \geq Y$, $Z+Y$ even, the corresponding Young tableau is given by

$$(a_1 a_2 \ldots a_{r-2} Y Z) \to (a_1 a_2 \ldots a_{r-2} \frac{Z-Y}{2} 00 \ldots) \,. \tag{10.66}$$

The Levi-Civita tensor can be used to convert long columns into short columns. For columns of r boxes, the Levi-Civita tensor splits $O(2r)$ reps into conjugate pairs of $SO(2r)$ reps.

We find the formula of King [98] and Murtaza and Rashid [115] the most convenient among various expressions for the dimensions of $SO(n)$ tensor reps given in the literature. If the Young tableau λ is represented as in sect. 9.3, the list of the row lengths $[\lambda_1, \lambda_2, \ldots \lambda_\kappa]$, then the dimension of the corresponding $SO(n)$ rep is given by

$$d_\lambda = \frac{d_S}{p!} \prod_{i=1}^{k} \frac{(\lambda_i + n - k - i - 1)!}{(n-2i)!} \prod_{j=1}^{k} (\lambda_i + \lambda_j + n - i - j) \,. \tag{10.67}$$

Here p is the total number of boxes, and d_S is the dimension of the symmetric group rep computed in (9.24). For $SO(2r)$ and $\kappa = r$, this rep is reducible and splits into a conjugate pair of reps. For example,

$$\begin{aligned}
\mathrm{d}_{\begin{array}{|c|c|}\hline \, & \, \\ \hline \, & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}} &= \frac{1}{\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}} \cdot (n+2)n(n-2) = \frac{n(n^2-4)}{3} \\
\mathrm{d}_{\begin{array}{|c|c|}\hline \, & \, \\ \hline \, & \multicolumn{1}{c}{} \\ \cline{1-1} \, & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}} &= \frac{(n+2)n(n-1)(n-3)}{8} \\
\mathrm{d}_{\begin{array}{|c|c|}\hline \, & \, \\ \hline \, & \, \\ \hline \end{array}} &= \frac{(n+2)(n+1)n(n-3)}{12} \,,
\end{aligned} \tag{10.68}$$

in agreement with (10.55). Even though the Dynkin labels distinguish $SO(2r+1)$ from $SO(2r)$ reps, this distinction is significant only for the spinor reps. The tensor reps of $SO(n)$ have the same Young tableaux for the even and the odd n's.

Chapter Eleven

Spinors

In chapter 10 we have discussed the tensor reps of orthogonal groups. However, the spinor reps of $SO(n)$ also play a fundamental role in physics, both as reps of space-time symmetries (Pauli spin matrices, Dirac gamma matrices, fermions in D-dimensional supergravities), and as reps of internal symmetries ($SO(10)$ grand unified theory, for example). In calculations of radiative corrections, the QED spin traces can easily run up to traces of products of some twelve gamma matrices [100], and efficient evaluation algorithms are of great practical importance. A most straightforward algorithm would evaluate such a trace in some $11!! = 11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \simeq 10,000$ steps. Even computers shirk such tedium. A good algorithm, such as the ones we shall describe here, will do the job in some $6^2 \simeq 100$ steps.

Spinors came to Cartan [20] as an unexpected fruit of his labors on the complete classification of reps of the simple Lie groups. Dirac [52] rediscovered them while looking for a linear version of the relativistic Klein-Gordon equation. He introduced matrices γ_μ which were required to satisfy

$$(p_0\gamma_0 + p_1\gamma_1 + \ldots)^2 = (p_0^2 - p_1^2 - p_2^2 - \ldots)\ . \tag{11.1}$$

For $n = 4$ he constructed γ's as $[4\times4]$ complex matrices. For $SO(2r)$ and $SO(2r+1)$ γ-matrices were constructed explicitly as $[2^r \times 2^r]$ complex matrices by Weyl and Brauer [156].

In the early days, such matrices were taken as a literal truth, and Klein and Nishina [101] are reputed to have computed their celebrated Quantum Electrodynamics cross-section by multiplying γ-matrices by hand. Every morning, day after day, they would multiply away explicit [4×4] γ_μ matrices and sum over μ's. In the afternoon, they would meet in the cafeteria of the Niels Bohr Institute to compare their results.

Nevertheless, all information that is actually needed for spin traces evaluation is contained in the Dirac algebraic condition (11.1), and today the Klein-Nishina trace over Dirac γ's is a textbook exercise, reducible by several applications of the Clifford algebra condition on γ-matrices

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}\,\mathbf{1}\ . \tag{11.2}$$

Iterative application of this condition immediately yields a spin traces evaluation algorithm in which the only residue of γ-matrices is the normalization factor $\mathrm{tr}\,\mathbf{1}$. However, this simple algorithm is inefficient in the sense that it requires a combinatorially large number of evaluation steps. The most efficient algorithm on the market (for any $SO(n)$) appears to be the one given by Kennedy [95, 44]. In Kennedy's algorithm, one views the spin trace to be evaluated as a $3n$-j coefficient.

Fierz [68] identities are used to express this $3n$-j coefficient in terms of 6-j coefficients (see sect. 11.3). Gamma matrices are $[2^{n/2}\times 2^{n/2}]$ in even dimensions, $[2^{(n-1)/2}\times 2^{(n-1)/2}]$ in odd dimensions, and at first sight it is not obvious that a smooth analytic continuation in dimension should be possible for spin traces. The reason why the Kennedy algorithm succeeds is that spinors are really not there at all. Their only role is to restrict the $SO(n)$ Clebsch-Gordan series to fully antisymmetric reps. The corresponding 3-j and 6-j coefficients are relatively simple combinatoric numbers, with analytic continuations in terms of gamma functions. The case of 4 spacetime dimensions is special because of the reducibility of $SO(4)$ to $SU(2) \otimes SU(2)$. Farrar and Neri [62], who as of April 18 1983 have computed in excess of 58,149 Feynman diagrams, have used this structure to develop a very efficient method for evaluating $SO(4)$ spinor expressions. An older technique, described here in sect. 11.8, is the Kahane [90] algorithm, which implements diagrammatically the Chisholm [27] identities. REDUCE, an algebra manipulation program written by Hearn [83], uses the Kahane algorithm. Thörnblad [150] has used $SO(4) \subset SO(5)$ embedding to speed-up evaluation of traces for massive fermions.

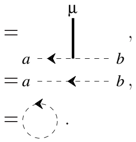
11.1 SPINOGRAPHY

Kennedy [95] introduced diagrammatic notation for γ-matrices

$(\gamma^\mu)_{ab} =$, $\quad a, b = 1, 2, \ldots, 2^{n/2}$ or $2^{(n-1)/2}$

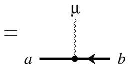
$\mathbf{1}_{ab} =$, $\quad \mu = 1, 2, \ldots, n$

$\mathrm{tr}\,\mathbf{1} =$. (11.3)

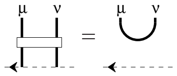


In this context, birdtracks go under the name "spinography". For notational simplicity, we take all γ-indices to be lower indices and omit arrows on the n-dimensional rep lines. The n-dimensional rep is drawn by a solid directed line to conform to the birdtrack notation of chapter 4. For QED and QCD spin traces, one might prefer the conventional Feynman diagram notation

$(\gamma^\mu)_{ab} =$

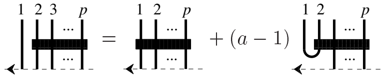


where the photons/gluons are in the n-dimensional rep of $SO(3,1)$, and electrons are spinors. We eschew such notation here, as it would conflict with $SO(n)$ birdtracks of chapter 10. The Clifford algebra anti-commutator condition (11.2) is given by



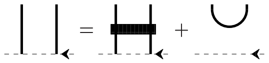
(11.4)

For antisymmetrized products of γ-matrices, this leads to the relation

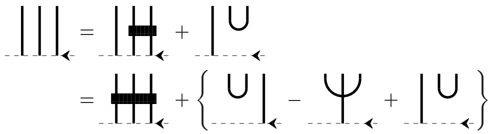


(11.5)

(we leave the proof as an exercise). Hence any product of γ-matrices can be expressed as a sum over antisymmetrized products of γ-matrices. For example, substitute the Young projection operators from table 9.1 into the products of two and three γ-matrices and use the Clifford algebra (11.4):



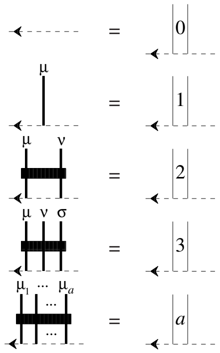
(11.6)



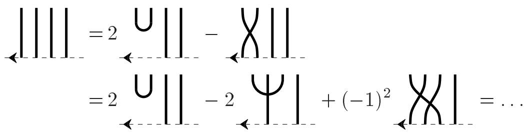
, *etc.* (11.7)

Only the fully antisymmetrized products of γ's are immune to reduction by (11.4). Hence, the antisymmetric tensors

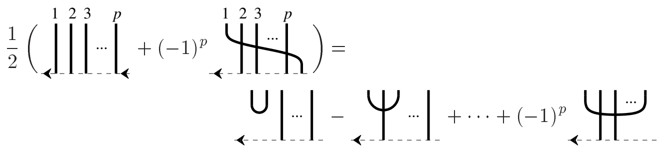
$$
\begin{aligned}
\Gamma^{(0)} &= \mathbf{1} \\
\Gamma^{(1)}_{\mu} &= \gamma_\mu \\
\Gamma^{(2)}_{\mu\nu} &= \tfrac{1}{2}[\gamma_\mu, \gamma_\nu] \\
\Gamma^{(3)}_{\mu\nu\sigma} &= \gamma_{[\mu}\gamma_\nu\gamma_{\sigma]} \\
\Gamma^{(a)}_{\mu_1\nu_2\ldots\mu_a} &= \gamma_{[\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_a]}
\end{aligned}
\tag{11.8}
$$



provide a complete basis for expanding products of γ-matrices. Applying the anticommutator (11.4) to a string of γ's, we can move the first γ all the way to the right and obtain



(11.9)



$$
\frac{1}{2}(\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_p} \pm \gamma^{\mu_2}\ldots\gamma^{\mu_p}\gamma^{\mu_1}) = g^{\mu_1\mu_2}\gamma^{\mu_3}\ldots\gamma^{\mu_p} - g^{\mu_1\mu_3}\gamma^{\mu_2}\gamma^{\mu_p} + \ldots \tag{11.10}
$$

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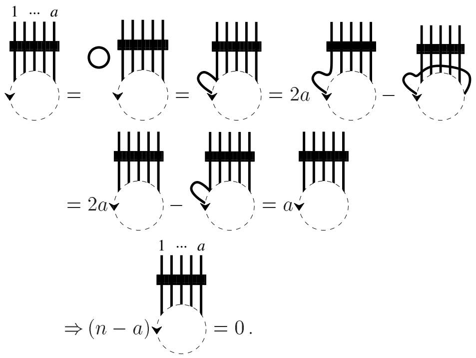
This identity has three immediate consequences

(i) traces of odd numbers of γ's vanish for n even

(ii) traces of even numbers of γ's can be evaluated recursively

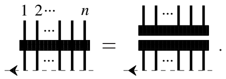
(iii) the result does not depend on the direction of the spinor line.

According to (11.10), any γ-matrix product can be expressed as a sum of terms involving $g_{\mu\nu}$'s and the antisymmetric basis tensors $\Gamma^{(a)}$, so in order to prove (i) we need only to consider traces of $\Gamma^{(a)}$ for a odd. This may be done as follows:



(11.11)

In the third step we have used (11.10) and the fact that a is odd. Hence, $\mathrm{tr}\,\Gamma^{(a)}$ vanishes for all odd a if n is even. If n is odd, $\mathrm{tr}\,\Gamma^{(n)}$ does not vanish because by (6.28)

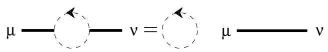


(11.12)

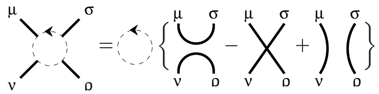
The n-dimensional analogue of the γ_5,

$$
\varepsilon^{\mu\nu\ldots\sigma}\gamma_\mu\gamma_\nu\ldots\gamma_\sigma \tag{11.13}
$$

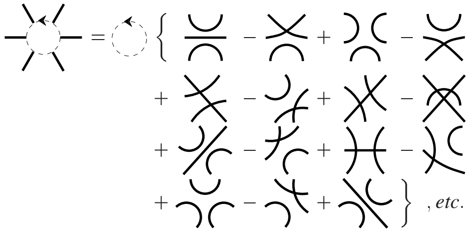
commutes with all γ-matrices, and, by Schur's lemma, it must be a multiple of the unit matrix, so it cannot be traceless. This proves (i). (11.10) relates traces of length p to traces of length $p-2$, so (ii) gives



$$
\mathrm{tr}\,\gamma_\mu\gamma_\nu = (\mathrm{tr}\,\mathbf{1})\,g_{\mu\nu}\,, \tag{11.14}
$$



$$
\mathrm{tr}\,\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = \mathrm{tr}\,\mathbf{1}\,\{g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} + g_{\mu\nu}g_{\nu\rho}\}\,, \tag{11.15}
$$

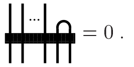


(11.16)

The result is always the $(2p-1)!!$ ways of pairing $2p$ indices with p Kronecker deltas. It is evident that nothing depends on the direction of spinor lines, as spinors are remembered only by an overall normalization factor $\mathrm{tr}\,\mathbf{1}$. The above identities are in principle a solution of the spinor traces evaluation problem. In practice they are intractable, as they yield a factorially growing number of terms in intermediate steps of trace evaluation.

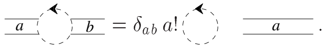
11.2 FIERZING AROUND

The algorithm (11.16) is too cumbersome for evaluation of traces of more than four or six γ-matrices. A more efficient algorithm is obtained by going to the Γ basis (11.8). Evaluation of traces of two and three Γ's is a simple combinatoric exercise using the expansion (11.16). Any term in which a pair of $g_{\mu\nu}$ indices gets antisymmetrized vanishes:



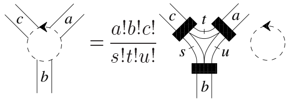
(11.17)

That implies that Γ's are *orthogonal*:



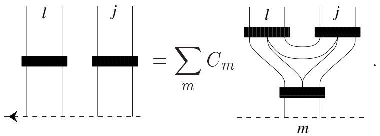
(11.18)

Here $a!$ is the number of terms in the expansion (11.16) which survive antisymmetrization (11.18). A trace of 3 Γ's is obtained in the same fashion



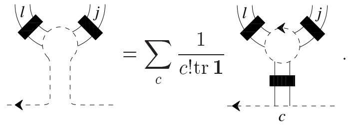
$$s=\frac{1}{2}(b+c-a)\,,\qquad t=\frac{1}{2}(c+a-b)\,,\qquad u=\frac{1}{2}(a+b-c)\,.$$

As Γ's provide a complete basis, we can express a product of two Γ matrices as a sum over Γ's, with extra indices carried by $g_{\mu\nu}$'s. From symmetry alone we know that terms in this expansion are of the form



(11.19)

The coefficients C_m can be computed by tracing both sides with Γ^c and using the orthogonality relation (11.18):

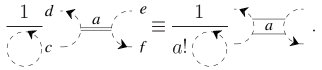


(11.20)

We do not have to consider traces of six or more Γ's, as they can all be reduced to three-Γ traces by the above relation.

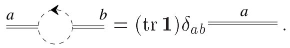
Let us now streamline the birdtracks. The orthogonality of Γ's (11.18) enables us to introduce projection operators

$$(P_a)_{cd,ef}=\frac{1}{a!\,\mathrm{tr}\,\mathbf{1}}\left(\gamma_{[\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_a]}\right)_{ab}\left(\gamma^{\mu_a}\ldots\gamma^{\mu_2}\gamma^{\mu_1}\right)_{cd}$$



(11.21)

The factor $\mathrm{tr}\,\mathbf{1}$ is a convenient (but inessential) normalization convention. It is analogous to the normalization factor a in (4.27):



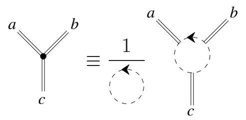
(11.22)

With this normalization, each spinor loop will carry factor $(\mathrm{tr}\,\mathbf{1})^{-1}$, and the final results will have no $\mathrm{tr}\,\mathbf{1}$ factors. k, j …are rep labels, not indices, and the repeated index summation convention does not apply. Only the fully antisymmetric $SO(n)$ reps occur, so a single integer (corresponding to the number of boxes in the single Young tableau column) is sufficient to characterize a rep.

For the trivial and the single γ-matrix reps, we shall omit the labels,

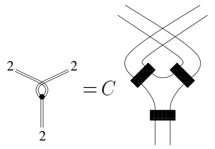
(11.23)

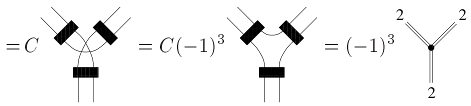
in keeping with the original definitions (11.3). The 3-Γ trace (11.19) defines a 3-vertex



(11.24)

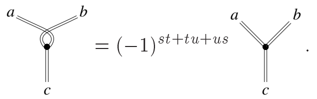
which is non-zero only if $a+b+c$ is even, and if a, b and c satisfy the triangle inequalities $|a-b|\leq c\leq|a+b|$. We appologize for using a, b, c both for the $SO(n)$ antisymmetric representations labels, and for spinor indices in (11.3), but the latin alphabet has only so many letters. It is important to note that in this definition the spinor loop runs anti-clockwise, as this vertex can change sign under interchange of two legs. For example, by (11.19)





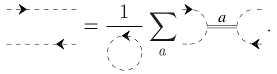
(11.25)

This vertex couples three adjoint representations (10.13) of $SO(n)$, and the sign rule is the usual rule (4.44) for the antisymmetry of C_{ijk} constants. The general sign rule follows from (11.19):



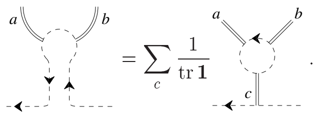
(11.26)

The projection operators P_a (11.21) satisfy the completeness relation (5.8):

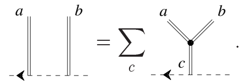


(11.27)

This follows from the completeness of Γ's, used in deriving (11.20). We have already drawn the left-hand side of (11.20) in such a way that the completeness relation (11.27) is evident:



In terms of the vertex (11.24) we get

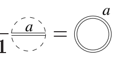


(11.28)

In this way we can systematically replace a string of γ-matrices by trees of 3-vertices.

Before moving on, let us check the completeness of P_a. P_a projects spinor $\otimes$ antispinor $\rightarrow$ antisymmetric a-index tensor rep of $SO(n)$. Its dimension was computed in (6.21):

$$d_a = \mathrm{tr}\, P_a = \frac{1}{\mathrm{tr}\,\mathbf{1}}$$



$$= \binom{n}{a} . \qquad (11.29)$$

d_a is automatically equal to zero for $n < a$; this guarantees the correctness of treating (11.29) as an arbitrarily large sum, even though for a given n it terminates at $a = n$. Tracing both sides of the completeness relation (11.27), we obtain a dimension sum rule:

$$(\mathrm{tr}\,\mathbf{1})^2 = \sum_a d_a = \sum_{a=0}^{n} \binom{n}{a} = (1+1)^n = 2^n . \qquad (11.30)$$

This confirms the results of Weyl and Brauer [156]: for even dimensions the number of components is 2^n, so Γ's can be represented by complex $[2^{n/2} \times 2^{n/2}]$ matrices. For odd dimensions there are two inequivalent spinor reps represented by $[2^{(n-1)/2} \times 2^{(n-1)/2}]$ matrices (see sect. 11.7). This inessential complication has no bearing on the evaluation algorithm we are about to describe.

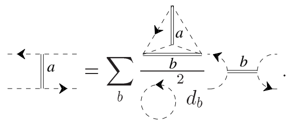
11.2.1 Exemplary evaluations

What have we accomplished? Iterating the completeness relation (11.28) we can make γ-matrices disappear altogether, and spin trace evaluation reduces to combinatorics of 3-vertices defined by the right-hand side of (11.19). This can be done, but is it any quicker than the simple algorithm (11.16)? The answer is yes: high efficiency can be achieved by viewing a complicated spin trace as a $3n$-j coefficient of sect. 5.2. To be concrete, take an eight γ-matrix trace as an example:

$$\mathrm{tr}\,(\gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta \gamma^\nu \gamma^\mu \gamma^\beta \gamma^\alpha) =$$

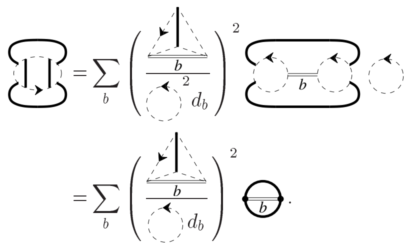
(11.31)

Such $3n$-j coefficient can be reduced by repeated application of the recoupling relation (5.13)



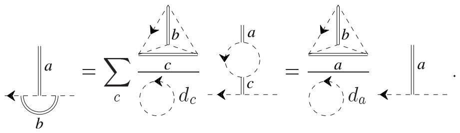
(11.32)

In the present context this relation is known as the Fierz identity [68]. It follows from two applications of the completeness relation, as in (5.13). Now we can redraw the 12-j coefficient from (11.31) and fierz on



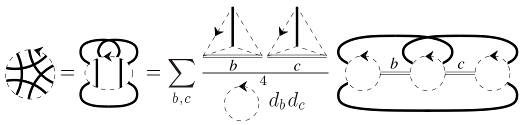
(11.33)

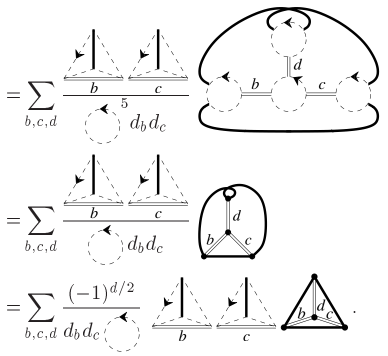
Another example is the reduction of a vertex diagram, a special case of the Wigner-Eckhart theorem (5.24):



(11.34)

As the final example we reduce a trace of 10 matrices:



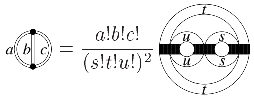


(11.35)

In this way, any spin trace can be reduced to a sum over 6-j and 3-j coefficients. Our next task is to evaluate these.

11.3 FIERZ COEFFICIENTS

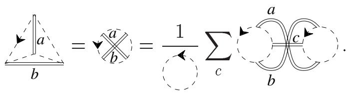
The 3-j coefficient in (11.33) can be evaluated by substituting (11.19) and doing "some" combinatorics



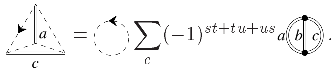
$$= \frac{a!b!c!}{(s!t!u!)^2} \quad = \frac{1}{s!t!u!}\frac{n!}{(n-s-t-u)!}\,. \qquad (11.36)$$

s, t, u are defined in (11.19). Note that $a+b+c = 2(s+t+u)$, and $a+b+c$ is even, otherwise the traces in the above formula vanish.

The 6-j coefficients in the Fierz identity (11.32) are not independent of the above 3-j coefficients. Redrawing a 6-j coefficient slightly, we can apply the completeness relation (11.28) to obtain:

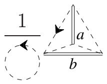


Interchanging j and k by the sign rule (11.26), we express the 6-j coefficient as a sum over 3-j coefficients:



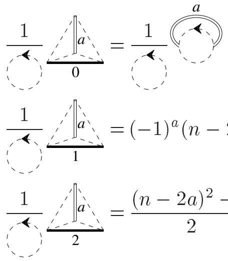
$$\sum_c (-1)^{st+tu+us} \qquad (11.37)$$

Using relations $t = a-u$, $s = b-u$, $a+t+u = a+b-u$, we can replace [22] the sum over c by the sum over u:



$$= (-1)^{ab}\binom{b}{n}\sum_u (-1)^u \binom{b}{u}\binom{n-b}{a-u}\,. \qquad (11.38)$$

u ranges from 0 to a or b, whichever is smaller, and the 6-j's for low values of a are particularly simple



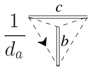
$$= d_a\,, \qquad (11.39)$$

$$= (-1)^a (n-2a) d_a\,, \qquad (11.40)$$

$$= \frac{(n-2a)^2 - n}{2} d_a\,. \qquad (11.41)$$

$$\vdots$$

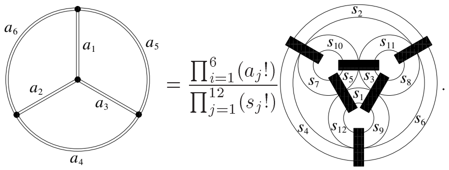
Kennedy [95] has tabulated Fierz coefficients F_{bc}, $b, c \le 6$. They are related to 6-j's by:



$$F_{bc} = \frac{b!}{c!}\frac{1}{d_a} \quad = (-1)^{bc}\frac{b!}{c!}\sum_{a=0}^{b}(-1)^u\binom{a}{u}\binom{n-a}{b-u}\,. \qquad (11.42)$$

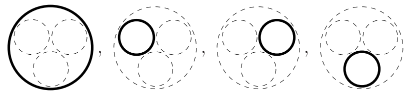
11.4 6-J COEFFICIENTS

To evaluate (11.35) we need 6-j coefficients for six antisymmetric tensor reps of $SO(n)$. Substitutions (11.24), (11.21) and (11.19) lead to a strand-network [132] expression for a 6-j coefficient



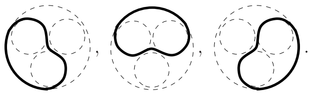
$$= \frac{\prod_{i=1}^{6}(a_j!)}{\prod_{j=1}^{12}(s_j!)} \qquad (11.43)$$

Pick out a line in a strand, and follow its possible routes through the strand network. Seven types of terms give non-vanishing contributions: 4 "mini tours"



(11.44)

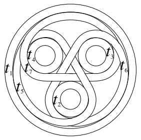
and 3 "grand tours"



(11.45)

Let the numbers of lines in different tours be $t_1, t_2, t_3, t_4, t_5, t_6$ and t_7. A non-vanishing contribution to the 6-j coefficient (11.43) corresponds to a partition of twelve strands, $s_1, s_2, \ldots, s_{12}$ into seven tours $t_1, t_2, \ldots, t_7$

$$M(t_1) =$$

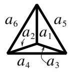


(11.46)

Comparing with (11.43), we see that each s_i is a sum of two t_i's: $s_1 = t_2 + t_7$, $s_2 = t_1 + t_7$, *etc*. It is sufficient to specify one t_1; this fixes all t_i's. Now one stares at the above figure and writes down

$$M(t_1) = \binom{n}{t} \frac{t!}{\prod_{i=1}^{7} t_i!} \frac{\prod_{i=1}^{12} s_i!}{\prod_{j=1}^{7} a_j!}, \qquad t = t_1 + t_2 + \ldots + t_7 \qquad (11.47)$$

(a well-known theorem states that combinatorial factors are impossible to explain [85]). The $\binom{n}{t}$ factor counts the number of ways of coloring $t_1 + t_2 + \ldots + t_7$ lines with n different colors. The second factor counts the number of distinct partitions of t lines into seven strands $t_1, t_2 \ldots, t_7$. The last factor again comes from the projector operator normalizations and the number of ways of coloring each strand and cancels against the corresponding factor in (11.43). Summing over the allowed partitions (for example, taking $0 \leq t_1 \leq s_2$), we finally obtain an expression for the 6-j coefficients:



$$= \sum_t \binom{n}{t} \frac{t!}{t_1! t_2! t_3! t_4! t_5! t_6! t_7!}$$

$$t_1 = -\frac{a_1 + a_2 + a_3}{2} + t \qquad t_5 = \frac{a_1 + a_3 + a_4 + a_6}{2} - t$$

$$t_2 = -\frac{a_1 + a_5 + a_6}{2} + t \qquad t_6 = \frac{a_1 + a_2 + a_4 + a_5}{2} - t$$

$$t_3 = -\frac{a_2 + a_4 + a_6}{2} + t \qquad t_7 = \frac{a_2 + a_3 + a_5 + a_6}{2} - t$$

$$t_4 = -\frac{a_3 + a_4 + a_5}{2} + t \,. \qquad (11.48)$$

The summation in (11.48) is over all values of t, such that all the t_i are non-negative integers. Naturally, the 3-j (11.36) is a special case of the 6-j (11.48). The 3-j's and 6-j's evaluated here, for all reps antisymmetric, should suffice in most applications.

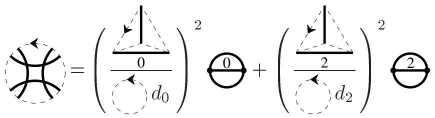
The above examples show how Kennedy's method produces the n-dimensional spinor reductions needed for the dimensional regularization [84]. Its efficiency for longer spin traces. Each γ-pair contraction produces one 6-j symbol, and the completeness relation sums do not exceed the number of pair contractions, so for $2p$ γ-matrices the evaluation does not exceed p^2 steps. This is far superior to the initial algorithm (11.16).

Finally, a learned comment to the wary of analytically continuing in n while relying on completeness sums (de Wit and 't Hooft [50, 142] anomalies). Trouble

could arise if, as we continued to low n, the $k > n$ terms in the completeness sum (11.27) gave non-vanishing contributions. We have explicitly noted that the dimension, 3-j and 6-j coefficients do vanish for any rep if $k > n$. The only danger arises from the Fierz coefficients (11.32): a ratio of 6-j and d can be finite for $j > n$. However, one is saved by the projection operator in the Fierz identity (11.32). This projection operator will eventually end up in some 6-j or 3-j coefficient without d in the denominator (like in (11.33)), and the whole term will vanish for $k > j$.

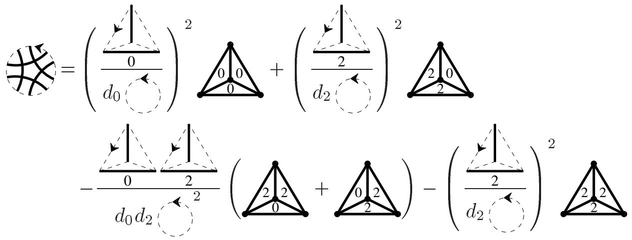
11.5 EXEMPLARY EVALUATIONS, CONTINUED

Now that we have explicit formulas for all 3-j and 6-j coefficients we can complete the evaluation of examples commenced in sect. 11.2.1. The eight γ-matrix trace (11.33) is given by



$$= n + n(n-1)(n-4)^2 \,, \qquad (11.49)$$

and the ten γ-matrix trace (11.35) by

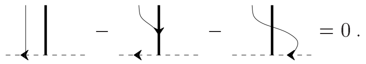


$$= n^3 + n(n-1)(n-4)^2 - 2n^2(n-1)(n-4)$$
$$-n(n-1)(n-2)(n-4)^2$$
$$= n^3 - n(n-1)(n-4)(n^2 - 5n + 12) \,. \qquad (11.50)$$

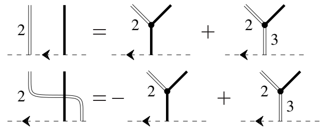
11.6 INVARIANCE OF γ-MATRICES

The above discussion of spinors did not follow the systematic approach of sect. 3.3 that we employ everywhere else in this monograph: start with a list of primitive invariants, find the characteristic equations that they satisfy, construct projection operators and identify the invariance group. In the present case, the primitive invariants are $g_{\mu\nu}$, δ_{ab} and $(\gamma_\mu)_{ab}$.We could retroactively construct the characteristic equation for $Q_{ab,cd} = (\gamma_\mu)_{ad}(\gamma_\mu)_{cb}$ from the Fierz identity (11.32), but the job is already done and the n eigenvalues are given by (11.38) - (11.41). The only thing that we still need to do is check that $SO(n)$, the invariance group of $g_{\mu\nu}$, is also the invariance group of $(\gamma_\mu)_{ab}$.

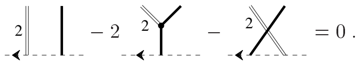
The $SO(n)$ Lie algebra is generated by the antisymmetric projection operator (8.7), or $\Gamma^{(2)}$ in the γ-matrix notation (11.8). The invariance condition (4.35) for γ-matrices is


(11.51)

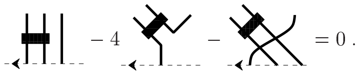
To check whether $\Gamma^{(2)}$ respects the invariance condition, we evaluate the first and the third term by means of the completeness relation (11.28):



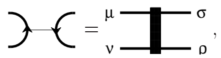
The minus sign comes from the sign rule (11.26). Subtracting, we obtain



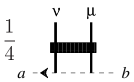
This already has the form of the invariance condition (11.51), modulo normalization convention. To fix the normalization, we go back to definitions (11.8), (11.24), (11.19):


(11.52)

The invariance condition (11.51) now fixes the relative normalizations of generators in the n-dimensional and spinor rep. If we take (8.7) for the n-dimensional rep

$(T_{\mu\nu})_{\rho\sigma} =$  (11.53)

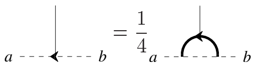
then the normalization of the generators in the spinor rep is

$(T_{\mu\nu})_{ab} = \frac{1}{4}$  $= \frac{1}{8}[\gamma_\nu, \gamma_\mu]$. (11.54)

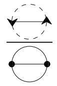
The γ-matrix invariance condition (11.51) written out in the tensor notation is

$$[T_{\mu\nu}, \gamma_\sigma] = \frac{1}{2}(g_{\mu\sigma}\gamma_\nu - g_{\nu\sigma}\gamma_\mu) \ . \tag{11.55}$$

If you prefer generators $(T_i)_{ab}$ indexed by the adjoint rep index $i = 1, 2, \ldots, N$, then you can use spinor rep generators defined as

$(T_i)_{ab} =$  (11.56)

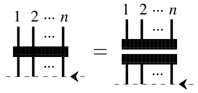
Now we can compute various casimirs for spinor reps. For example, the Dynkin index, sect. 7.4, for the lowest dimensional spinor rep is given by

$\ell =$  $= \frac{\mathrm{tr}\,\mathbf{1}}{8(n-2)} = \frac{2^{[\frac{n}{2}]-3}}{n-2}$. (11.57)

From the invariance of γ_μ follows invariance of all $\Gamma^{(k)}$. In particular, the invariance condition for $\Gamma^{(2)}$ is the usual Lie algebra condition (4.45) with the structure constants given by (11.25).

11.7 HANDEDNESS

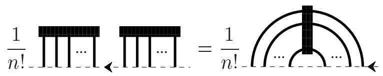
Among the bases (11.8), $\Gamma^{(n)}_{\mu_1\mu_2\ldots\mu_n}$ plays a special role; it projects onto a one-dimensional space, and the antisymmetrization can be replaced by a pair of Levi-Civita tensors (6.28)

$\Gamma^{(n)} =$  . (11.58)

The corresponding clebsches are the generalized "γ_5" matrices

$\gamma^* \equiv \frac{1}{\sqrt{n!}}$  $= i^{n(n-1)/2}\gamma_1\gamma_2\ldots\gamma_n$. (11.59)

The phase factor is, as explained in sect. 4.7, only a nuisance which cancels away in physical calculations. γ^* satisfies a trivial characteristic equation (use (6.28) and (11.18) to evaluate this)

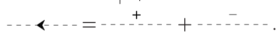
$(\gamma^*)^2 = \frac{1}{n!}$  $= \mathbf{1}$, (11.60)

which yields projection operators (4.16)

$$P_+ = \frac{1}{2}(\mathbf{1} + \gamma^*) \ , \qquad P_- = \frac{1}{2}(\mathbf{1} - \gamma^*) \ . \tag{11.61}$$

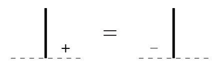
The reducibility of Dirac spinors does not affect the correctness of the Kennedy spin traces algorithm. However, as the reduction of Dirac spinors is of physical importance, we shall briefly describe the irreducible spinor reps. Let us denote the two projectors diagrammatically by

$$\mathbf{1} = P_+ + P_-$$

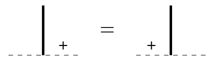
 (11.62)

In even dimensions $\gamma_\mu\gamma^* = -\gamma^*\gamma_\mu$, while in odd dimensions $\gamma_\mu\gamma^* = \gamma^*\gamma_\mu$, so

$$n \text{ even}: \begin{cases} \gamma_\mu P_+ = P_-\gamma_\mu \end{cases}$$

 , (11.63)

$$n \text{ odd:} \quad \gamma_\mu P_+ = P_+\gamma_\mu$$



(11.64)

Hence, in the odd dimensions Dirac γ_μ matrices decompose into a pair of conjugate $[2^{(n-1)/2}\times 2^{(n-1)/2}]$ reps:

$$n \text{ odd:} \qquad \gamma_\mu = P_+\gamma_\mu P_+ + P_-\gamma_\mu P_- \,, \tag{11.65}$$

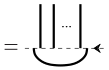
and the irreducible spinor reps are of dimension $2^{(n-1)/2}$.

11.8 KAHANE ALGORITHM

For the case of 4 dimensions, there is a fast algorithm for trace evaluation, due to Kahane [90].

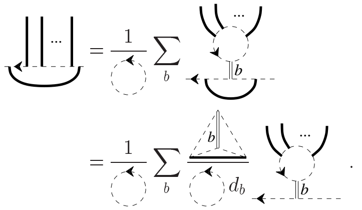
Consider a γ-matrix contraction

$$\gamma^a\gamma_b\gamma_c\cdots\gamma_d\gamma_a =$$



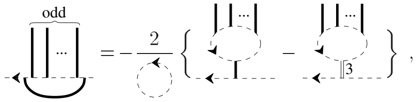
(11.66)

and use the completeness relation (11.27) and the "vertex" formula (11.34):

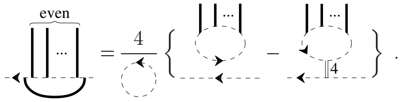


(11.67)

For $n = 4$, this sum ranges over $k = 0, 1, 2, 3, 4$. A spinor trace is non-vanishing only for even numbers of γ's, (11.16), so we distinguish the even and the odd cases when substituting the Fierz coefficients (11.40):

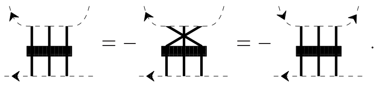


(11.68)



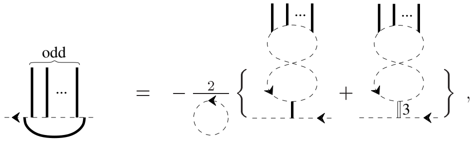
(11.69)

The sign of the second term in (11.68) can be reversed by transposing the 3 γ's (remember, the arrows on the spinor lines keep track of signs, cf. (11.24) and (11.26))



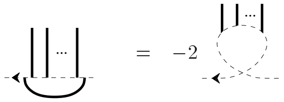
(11.70)

But now the term in the brackets in (11.68) is just the completeness sum (11.27), and the summation can be dropped:



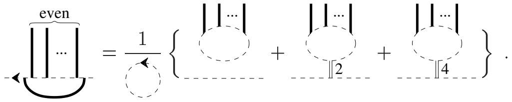
(11.71)

rule 1 :



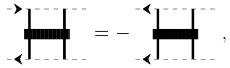
$$\gamma^a\gamma_b\gamma_c\cdots\gamma_d\gamma_a = -2\,\gamma_d\cdots\gamma_c\gamma_b$$

The same trick does not work for (11.69), because there the completeness sum has 3 terms:



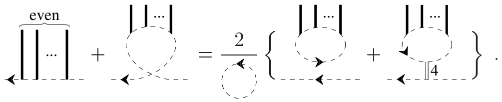
(11.72)

However, as $\gamma_{[a}\gamma_{b]} = -\gamma_{[b}\gamma_{a]}$



(11.73)

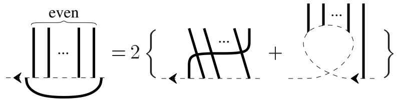
the sum of $\gamma_a\gamma_b\ldots\gamma_d$ and its transpose $\gamma_d\ldots\gamma_b\gamma_a$ has a two-term completeness sum:



(11.74)

Finally, we can change the sign of the second term in (11.69) by using $\{\gamma_5, \gamma_a\} = 0$;

rule 2 :



$$\gamma^e\gamma_a\gamma_b\ldots\gamma_c\gamma_d = 2\{\gamma_d\gamma_a\gamma_b\ldots\gamma_b\gamma_a\gamma_d\}\,. \tag{11.75}$$

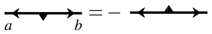
This rule and the rule (11.71) enable us to remove γ-contractions ("internal photon lines") one by one, at most doubling the number of terms at each step.These rules are special to $n = 4$ and have no n-dimensional generalization.

Chapter Twelve

Symplectic groups

Symplectic group $Sp(n)$ is the group of all transformations which leave invariant a skew symmetric quadratic form $(p,q) = f_{ab}p^a q^b$:

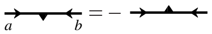
$$f_{ab} = -\, f_{ba} \qquad\qquad a, b = 1, 2, \ldots n$$



n even. (12.1)

The birdtrack notation is motivated by the need to distinguish the first and the second index: it is a special case of the birdtracks for antisymmetric tensors of even rank (6.57). If (p,q) is an invariant, so is its complex conjugate $(p,q)^* = f^{ba}p_a q_b$, and

$$f^{ab} = -\, f^{ba}$$



(12.2)

is also an invariant tensor. Matrix $A_a^b = f_{ac}f^{cb}$ must be proportional to unity, as otherwise its characteristic equation would decompose the defining n-dimensional rep. A convenient normalization is

$$f_{ac}f^{cb} = -\, \delta_a^b$$

(12.3)

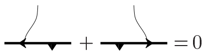
Indices can be raised and lowered at will, so the arrows on lines can be dropped. However, omitting symplectic warts (the black half-circles) appears perilous, as without them it is hard to keep track of signs. Our convention will be to perform all contractions with f^{ab} and omit the arrows but not the warts:

$f^{ab} =$  . (12.4)

All other tensors will have lower indices. The Lie group generators $(T_i)_a{}^b$ will be replaced by

$(T_i)_{ab} =$  . (12.5)

The invariance condition (4.35) for the symplectic invariant tensor is



$$(T_i)_{ac}f_{cb} + f_{ac}(T_i)_{cb} = 0\,. \qquad (12.6)$$

A skew symmetric matrix f_{ab} has the inverse in (12.3) only if $\det f \neq 0$. That is possible only in even dimensions, so $Sp(n)$ can be realized only for even n.

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In this chapter we shall outline the construction of $Sp(n)$ tensor reps. They are obtained by contracting the irreducible tensors of $SU(n)$ with the symplectic metric f^{ab} and decomposing them into traces and traceless parts. The representation theory for $Sp(n)$ is analogous in step-by-step fashion to the representation theory for $SO(n)$. This arises, because the two groups are related by supersymmetry, and in chapter 13 we shall exploit this connection by showing, that all group-theoretic weights for the two groups are related by analytic continuation into negative dimensions.

12.1 TWO-INDEX TENSORS

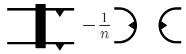
The decomposition goes the same way as for $SO(n)$, sect. 10.1. The matrix (10.8), given by

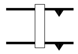
$T =$  , (12.7)

satisfies the same characteristic equation (10.9) as for $SO(n)$. Now T is antisymmetric, $AT = T$, and only the antisymmetric subspace gets decomposed. $Sp(n)$ 2-index tensors decompose is

singlet: $(P_1)_{ab,cd} = \frac{1}{n} f_{ab}f_{cd} = \frac{1}{n}$ 

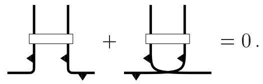
antisymmetric: $(P_2)_{ab,cd} = \frac{1}{2}(f_{ad}f_{bc} - f_{ac}f_{bd}) - \frac{1}{n}f_{ab}f_{cd}$

$=$ 

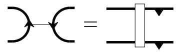
symmetric: $(P_3)_{ab,cd} = \frac{1}{2}(f_{ad}f_{bc} + f_{ac}f_{bd}) =$  .

(12.8)

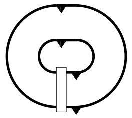
The $SU(n)$ adjoint rep (10.14) is now split into traceless symmetric and antisymmetric parts. The adjoint rep of $Sp(n)$ is given by the symmetric subspace, as only P_3 satisfies the invariance condition (12.6):



Hence, the adjoint rep projection operator for $Sp(n)$ is given by

$\frac{1}{a}$  . (12.9)

The dimension of $Sp(n)$ is

$N = \mathrm{tr}\, P_A =$  $= \frac{n(n+1)}{2}\,.$ (12.10)

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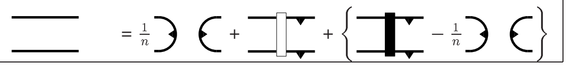
Young tableaux	$\square \otimes \square$	=	$\bullet$	+	$\square\square$	+	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$
Dynkin labels	$(10\ldots)\times(10\ldots)$	=	$(00\ldots)$	+	$(010\ldots)$	+	$(20\ldots)$
Dimensions	n^2	=	1	+	$\frac{n(n+1)}{2}$	+	$\frac{(n-2)(n+1)}{2}$
Dynkin indices	$2n\frac{1}{n+2}$	=	0	+	1	+	$\frac{n-2}{n+2}$
Projectors							

Table 12.1 $Sp(n)$ Clebsch-Gordan series for $V \otimes V$.

Remember that all contractions are carried out by f^{ab} - hence, the extra warts in the trace expression.

Dimensions of the other reps and the Dynkin indices (see sect. 7.4) are listed in table 12.1.

Chapter Thirteen

Negative dimensions

A cursory examination of the expressions for the dimensions and the Dynkin indices listed in tables 7.4 and 7.6, and in the tables of chapter 9, chapter 10 and chapter 12, reveals intriguing symmetries under substitution $n \to -n$. This kind of symmetry is best illustrated by the reps of $SU(n)$; if λ stands for a Young tableau with p boxes, and $\overline{\lambda}$ for the transposed tableau obtained by flipping λ across the diagonal (*ie.*, exchanging symmetrizations and antisymmetrizations), then the dimensions of the two tableaux are related by

$$SU(n): \qquad d_\lambda(n) = (-1)^p d_{\overline{\lambda}}(-n) \; . \tag{13.1}$$

This is evident from the standard recipe for computing the $SU(n)$ rep dimensions, sect. 9.3, as well as from the expressions listed in the tables of chapter 9. In all cases, exchanging symmetrizations and antisymmetrizations amounts to replacing n by $-n$.

Such relations have been noticed before; Parisi and Sourlas [127] have suggested, that a Grassmann vector space of dimension n can be interpreted as an ordinary vector space of dimension $-n$. Penrose [132] has introduced the term "negative dimensions" in his construction of $SU(2) \simeq Sp(2)$ reps as $SO(-2)$. King [98] has proved that the dimension of any irreducible rep of $Sp(n)$ is equal to that of $SO(n)$ with symmetrizations exchanged with antisymmetrizations (*ie.* corresponding to the transposed Young tableau), and n replaced by $-n$. Mkrtchyan [114] has observed this relation for the QCD loop equations. With the advent of supersymmetries, $n \to -n$ relations have become commonplace, as they are built into the structure of groups such as the orthosympletic group $OSp(b, f)$. Some highly nontrivial examples of $n \to -n$ symmetries for the exceptional groups [41] will be discussed in chapter 20.

Here, we shall prove the following:

Theorem 1. For any $SU(n)$ invariant scalar exchanging symmetrizations and antisymmetrizations is equivalent to replacing n by $-n$:

$$SU(n) = \overline{SU}(-n) \; . \tag{13.2}$$

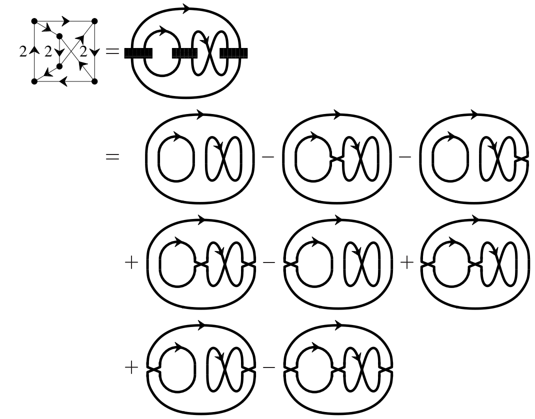
Theorem 2. For any $SO(n)$ invariant scalar there exists the corresponding $Sp(n)$ invariant scalar (and vice versa), obtained by exchanging symmetrizations and antisymmetrizations, replacing the $SO(n)$ symmetric bilinear invariant δ_{ab} by the $Sp(n)$ antisymmetric bilinear invariant f_{ab}, and replacing n by $-n$:

$$SO(n) = \overline{Sp}(-n) \; , \qquad Sp(n) = \overline{SO}(-n) \tag{13.3}$$

(the bars on $\overline{SU}$, $\overline{Sp}$, $\overline{SO}$ indicate transposition, *ie.* exchange of symmetrizations and antisymmetrizations).

Various examples of $n \to -n$ relations, which we now give, cited in literature, are all special cases of these general theorems whose proof is much simpler than the published proofs for the special cases.

As we have argued in sect. 5.2, all physical consequences of a symmetry (rep dimensions, level splittings, *etc*) can be expressed in terms of invariant scalars (the equivalence (13.2), (13.3) for arbitrary scalar invariants). The idea of the proof is illustrated by the following typical computation: evaluate, for example, the following $SU(n)$ 9-j coefficient for recoupling of three antisymmetric rank-2 reps:



$$\begin{aligned} &= n^3 - n^2 - n^2 + n - n^2 + n + n - n^2 \\ &= n(n-1)(n-3) \; . \end{aligned} \tag{13.4}$$

Notice that in the expansion of the symmetry operators, the graphs with an odd number of crossings give an even power of n, and vice versa. If we change the three symmetrizers into antisymmetrizers, the terms, which change the sign, are exactly those with an even number of crossings. The crossing in the original graph, which had nothing to do with any symmetry operator, appears in every term of the expansion, and this does not affect our conclusion; an exchange of symmetrizations and antisymmetrizations amounts to substitution $n \to -n$. The overall sign is only a matter of convention; it depends on how we define vertices in $3n$-j's. The proof for the general $SU(n)$ case is even simpler than the above example:

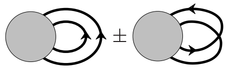
13.1 $SU(N) = \overline{SU}(-N)$

The primitive invariant tensors of $SU(n)$ are the Kronecker tensor δ^a_b and the Levi-Civita tensor $\varepsilon_{a_1 \cdots a_n}$. All other invariants of $SU(n)$ are built from these two objects.

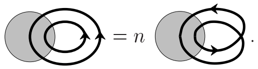
A scalar ($3n$-j coefficient, vacuum bubble) is a number which, in birdtrack notation, corresponds to a graph with no external legs.

As the directed lines must end somewhere, the Levi-Civita tensors can be present only in pairs and can always be eliminated by the identity (6.28). An $SU(n)$ $3n$-j coefficient, therefore, corresponds to a diagram made solely of closed loops of directed lines and symmetry projection operators, like the example (13.4).

Consider the graph corresponding to an arbitrary $SU(n)$ scalar, and expand all its symmetry operators as in (13.4). The expansion can be arranged (in any of many possible ways) as a sum of pairs of form

$\ldots +$  $+ \ldots$ (13.5)

with a plus sign if the crossing arises from a symmetrization, and a minus sign if it arises from an antisymmetrization. Each graph consists only of closed loops, *ie.* a definite power of n, and thus uncrossing two lines can have one of two consequences. If the two crossed line segments come from the same loop, then uncrossing splits this into two loops, whereas if they come from two loops, it joins them into one loop. The power of n is changed by the uncrossing:

 (13.6)

Hence, the pairs in the expansion (13.5) always differ by $n^{\pm 1}$, and exchanging symmetrizations and antisymmetrizations has the same effect as substituting $n \to -n$ (up to an irrelevant overall sign). This completes the proof of (13.2).

Some examples of $n \to -n$ relations for $SU(n)$ reps:

(i) Dimensions of the fully symmetric reps (6.13) and the fully antisymmetric reps (6.21) are related by the gamma-function analytic continuation formula

$$\frac{n!}{(n-p)!} = (-1)^p \frac{(-n+p-1)!}{(-n-1)!} \,. \qquad (13.7)$$

(ii) The reps (7.20) and (7.21) correspond to the 2-index symmetric, antisymmetric tensors, respectively. Therefore, their dimensions in table 9.1 are related by $n \to -n$.

(iii) The reps (7.44) and (7.45), see also table 7.6, are related by $n \to -n$ for the same reason.

(iv) $n \to -n$ symmetries in table 7.2.

(v) Dimension formula (13.1).

13.2 $SO(N) = \overline{SP}(-N)$

In addition to δ^a_b and $\varepsilon_{ab\ldots d}$, $SO(n)$ preserves a symmetric bilinear invariant δ_{ab}, for which we have introduced birdtrack open circle notation in (8.1). Such open circles

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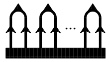
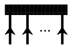
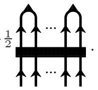
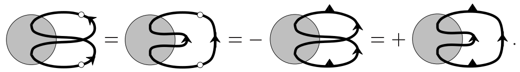
can occur in $SO(n)$ $3n$-j graphs, flipping the line directions. The Levi-Civita tensor still cannot occur, as directed lines, starting on an ε tensor, would have to end on a d tensor, which gives zero by symmetry. $Sp(n)$ differs from $SO(n)$ by having a skew symmetric bilinear tensor f_{ab}, for which we have introduced birdtrack wart notation in (12.1). A Levi-Civita tensor can appear in an $Sp(n)$ $3n$-j graph, but as

 $= \sqrt{\det f}$, (13.8)

(an exercise for the reader), a Levi-Civita can always be replaced by an antisymmetrization

 $= (\det f)^{-\frac{1}{2}}$  . (13.9)

For any $SO(n)$ scalar there exists a corresponding $Sp(n)$ scalar, obtained by exchanging the symmetrizations and antisymmetrizations *and* the δ_{ab}'s and f_{ab}'s in the corresponding graphs. The proof that the two scalars are transformed into each other by replacing n by $-n$, is the same as for $SU(n)$, except that the two line segments at a crossing could come from a new kind of loop, containing δ_{ab}'s or f_{ab}'s. In that case, equation (13.6) is replaced by

 (13.10)

While now uncrossing the lines does not change the number of loops, changing δ_{ab}'s to f_{ab}'s does provide the necessary minus sign. This completes the proof of (13.3) for the tensor reps of $SO(n)$ and $Sp(n)$.

Some examples of $SO(n) = \overline{Sp}(-n)$ relations:

(i) The $SO(n)$ antisymmetric adjoint rep (10.13) corresponds to the $Sp(n)$ symmetric adjoint rep (12.9).

(ii) Compare table **??** and table 10.1.

(iii) Penrose [132] binors: $SU(2) = \overline{SO}(-2)$.

In order to extend the proof to the spinor reps, we will first have to invent the $Sp(n)$ analog of spinor reps. We turn to this task in the next chapter.

Chapter Fourteen

Spinors' $Sp(n)$ sisters

Dirac discovered spinors in his search for a vectorial quantity which could be interpreted as a "square root" of the Minkowski 4-momentum squared,

$$(p_4\gamma_4 + p_3\gamma_3 + \ldots)^2 = p_4^2 - p_3^2 - p_2^2 - \ldots .$$

What happens if one extends a Minkowski 4-momentum (p_4, p_3, p_2, p_1) into fermionic, Grassmann dimensions $(p_4, p_3, p_2, p_1, p_{-1}, p_{-2}, p_{-3}, \cdots, p_{-n})$? The Grassmann sector p_i anticommute, and the gamma matrix relatives in the Grassmann dimensions have to satisfy the Heisenberg algebra commutation relation

$$[\gamma_i, \gamma_j] = f_{ij}\mathbf{1} ,$$

instead of the Clifford algebra anticommutator condition (11.2), with the bilinear invariant $f_{ij} = -f_{ji}$ skew-symmetric in the Grassmann dimensions.

In chapter 12, we showed that the symplectic group $Sp(n)$ is the invariance group of a skew-symmetric bilinear invariant f_{ij}. In sect. 14.1, we investigate the consequences of taking γ matrices to be Grassmann valued. We are led to a new family of objects, which we call spinsters. Spinsters play a role for symplectic groups analogous, to that played by spinors for orthogonal groups. With the aid of spinsters, we are able to compute, for example, all the 3-j and 6-j coefficients for symmetric reps of $Sp(n)$. We find that these coefficients are identical with those obtained for $SO(n)$, if we interchange the roles of symmetrization and antisymmetrization and simultaneously replace the dimension n by $-n$. In sect. 14.2, we make use of the fact that $Sp(2) \simeq SU(2)$, to show that the formulas for $SU(2)$ 3-j and 6-j coefficients are special cases of general expressions for these quantities, we derived earlier.

This chapter is based on ref. [44].

14.1 SPINSTERS

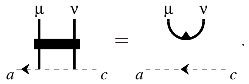
The Clifford algebra (11.2) Dirac matrix elements $(\gamma_\mu)_{ab}$ are commuting numbers. In this section we shall investigate consequences of taking γ_μ to be Grassmann valued.

$$(\gamma_\mu)_{ab}(\gamma_\nu)_{cd} = -(\gamma_\nu)_{cd}(\gamma_\mu)_{ab} . \qquad (14.1)$$

The Grassmann extension of the Clifford algebra (11.2) is

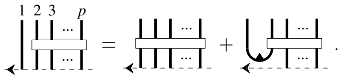
$$\frac{1}{2}[\gamma_\mu, \gamma_\nu] = f_{\mu\nu}\mathbf{1} , \qquad \mu, \nu = 1, 2, \ldots, n, \quad n \text{ even} . \qquad (14.2)$$

The anticommutator gets replaced by a commutator, and the $SO(n)$ symmetric invariant tensor $g_{\mu\nu}$ by the $Sp(n)$ skew-symmetric invariant tensor $f_{\mu\nu}$. Just as the Dirac gamma matrices lead to spinor reps of $SO(n)$, the Grassmann valued γ_μ give rise to $Sp(n)$ reps, which we shall call spinsters. Following the $Sp(n)$ diagramatic notation for the skew symmetric invariant tensor (12.1), we represent the defining commutation relation (14.2) by



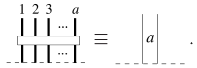
(14.3)

For the symmetrized products of γ matrices the above commutation relations leads to



(14.4)

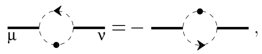
As in chapter 11, this gives rise to a complete basis for expanding products of γ matrices. Γ's are now the symmetrized products of γ matrices:



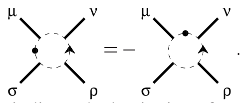
(14.5)

Note that while for spinors the $\Gamma^{(k)}$ vanish by antisymmetry for $k > n$, for spinsters the $\Gamma^{(k)}$'s are non-vanishing for any k, and the number of spinster basis tensors is infinite. However, a reduction of a product of k γ matrices involves only a finite number of $\Gamma^{(l)}$, $0 \leqslant l \leqslant k$. As the components $(\gamma_\mu)_{ab}$ are Grassmann valued, spinster traces of even numbers of γ's are anticyclic

$$\mathrm{tr}\, \gamma_\mu\gamma_\nu = (\gamma_\mu)_{ab}(\gamma_\nu)_{ba} = -\,\mathrm{tr}\, \gamma_\nu\gamma_\mu$$

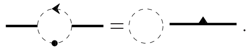


$$\mathrm{tr}\, \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = -\,\mathrm{tr}\, \gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu \qquad (14.6)$$



In the diagrammatic notation we indicate the beginning of a spinster trace by a dot. The dot keeps track of the signs in the same way as the wart (12.3) for $f_{\mu\nu}$. Indeed, tracing (14.3) we have

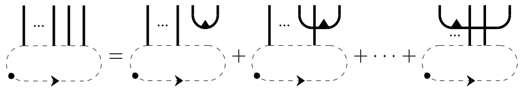
$$\mathrm{tr}\, \gamma_\mu\gamma_\nu = f_{\mu\nu}\, \mathrm{tr}\, \mathbf{1}$$



(14.7)

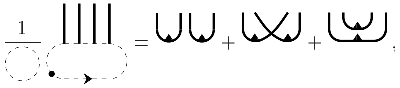
Moving a dot through a γ matrix gives a factor -1, as in (14.6).

Spinster traces can be evaluated recursively, as in (11.7). For a trace of an even number of γ's we have



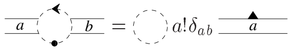
(14.8)

A trace of an odd number of γ's vanishes [44]. Iteration of the equation (14.8) expresses a spinster trace as a sum of the $(p-1)!! = (p-1)(p-3)\ldots 5.3.1$ ways of connecting the external legs with $f_{\mu\nu}$. The overall sign is fixed uniquely by the position of the dot on the spinster trace:

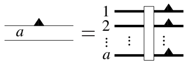

(14.9)

and so on (see (11.15)).

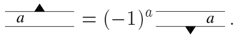
Evaluation of traces of several Γ's is again a simple combinatoric exercise. Any term in which a pair of $f_{\mu\nu}$ indices are symmetrized vanishes, which implies that any $\Gamma^{(k)}$ with $k > 0$ is traceless. The Γ's are orthogonal:


(14.10)

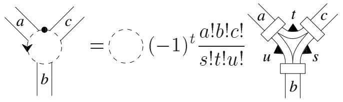
The symmetrized product of a $f_{\mu\nu}$'s denoted by


(14.11)

is either symmetric or skew-symmetric


(14.12)

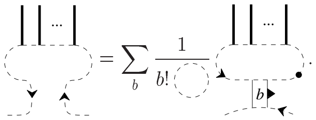
A spinster trace of three symmetric $Sp(n)$ reps defines a 3-vertex:



$$= 0 \quad \text{for } a+b+c = \text{odd},$$

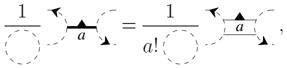
$$s = \frac{1}{2}(b+c-a), \quad t = \frac{1}{2}(c+a-b), \quad u = \frac{1}{2}(a+b-c). \quad (14.13)$$

As in (11.20), Γ's provide a complete basis for expanding products of arbitrary numbers of γ matrices


(14.14)

The coupling coefficients in (14.14) are computed as spinster traces using the orthogonality relation (14.10). As only traces of even numbers of γ's are nonvanishing, spinster traces are even Grassmann elements, they thus commute with any other Γ, and all the signs in the above completeness relation are unambiguous.

The orthogonality of Γ's enables us to introduce projection operators and 3-vertices


(14.15)

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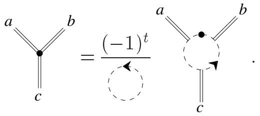
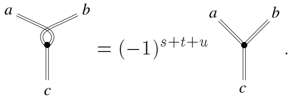
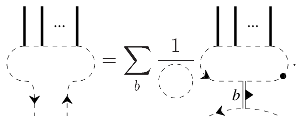

(14.16)

The sign factor $(-1)^t$ gives a symmetric definition of the 3-vertex, see (3.8). It is important to note that the spinster loop runs clockwise in this definition. Because of (3.38), the 3-vertex has a non-trivial symmetry under interchange of two legs:


(14.17)

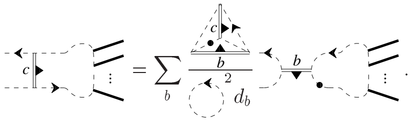
Note that this is different from (11.26) - one of the few instances of spinsters and spinors differing in a way which cannot be immediately understood as an $n \to -n$ continuation.

The completeness relation (14.14) can be written

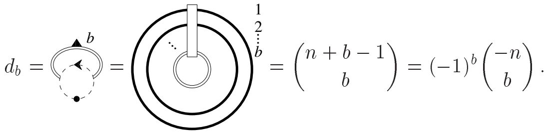

(14.18)

We keep an arbitrary number of γ's to indicate the way in which the spinster trace is to be taken; this keeps track of Grassmann signs.

The recoupling relation is derived as in the spinor case (11.32)

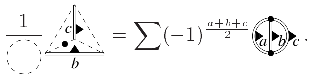

(14.19)

Here d_b is the dimension of the fully symmetrized b-index tensor rep of $Sp(n)$:


(14.20)

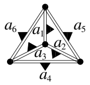
The spinster recoupling coefficients in (11.34) are analogues of the spinor Fierz coefficients in (11.32). Completeness can be used to evaluate spinster traces in the same way as in examples (11.34) to (11.35).

The next step is the evaluation of 3-j's, 6-j's and spinster recoupling coefficients. The spinster recoupling coefficients can be expressed in terms of 3-j's just as in (11.37):


(14.21)

The evaluation of 3-j and 6-j coefficients is again a matter of simple combinatorics:

 $$= (-1)^{s+t+u}\binom{n+s+t+u-1}{s+t+u}\frac{(s+t+u)!}{s!t!u!}\,, \qquad (14.22)$$

 $$= \sum_t \binom{n+t-1}{t}\frac{(-1)^t t!}{t_1!t_2!t_3!t_4!t_5!t_6!t_7!}\,, \qquad (14.23)$$

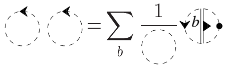
with the t_i defined in (11.48).

We close this section by a comment on the dimensionality of spinster reps. Tracing both sides of the spinor completeness relation (11.27) we determine the dimensionality of spinor reps from the sum rule (11.30)

$$(\mathrm{tr}\,\mathbf{1})^2 = \sum_{a=0}^{n}\binom{n}{a} = 2^n\,.$$

Hence, Dirac matrices (in even dimensions) are $[2^{n/2}\times 2^{n/2}]$, and the range of spinor indices in (11.3) is $a, b = 1, 2, \ldots, 2^{n/2}$.

For spinsters tracing the completeness relation (14.18) yields (the string of γ matrices was indicated only to keep track of signs for odd b's):

 $$= \sum_b d_b \qquad (14.24)$$

$$(\mathrm{tr}\,\mathbf{1})^2 = \sum_{c=o}^{\infty}\binom{n+b-1}{b}\,.$$

The spinster trace is infinite. This is the reason why spinster traces are not to be found in the list of the finite-dimensional irreducible reps of $Sp(n)$. One way of making the traces meaningful is to note that in any spinster trace evaluation only a finite number of Γ's are needed, so we can truncate the completeness relation (14.18) to terms $0 \leqslant b \leqslant b_{max}$. A more pragmatic attitude is to observe that the final results of the calculation are the 3-j and 6-j coefficients for the fully symmetric reps of $Sp(n)$, and that the spinster algebra (14.2) is a formal device for projecting only the fully symmetric reps from various Clebsch-Gordan series for $Sp(n)$.

The most striking result of this section is that the 3-j and 6-j coefficients are just the $SO(n)$ coefficients evaluated for $n \to -n$. The reason for this we already understand from chapter 13.

When we took the Grassmann extension of Clifford algebras in (14.2), it was not too surprising that the main effect was to interchange the role of symmetrization and antisymmetrization. All antisymmetric tensor reps of $SO(n)$ correspond to the symmetric rep of $Sp(n)$. What is more surprising is that if we take the expression we derived for the $SO(n)$ 3-j and 6-j coefficients and replace the dimension n by $-n$, we obtain exactly the corresponding result for $Sp(n)$. The negative dimension arises in these cases through the relation $\binom{-n}{a} = (-1)^a\binom{n+a-1}{a}$, which may be justified by analytic continuation of binomial coefficients by the Beta function.

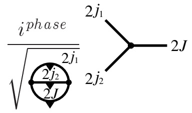
14.2 RACAH COEFFICIENTS

So far, we have computed the 6-j coefficients for fully symmetric reps of $Sp(n)$. $Sp(2)$ plays a special role here; the skew symmetric invariant $f^{\mu\nu}$ has only one independent component, and it must be proportional to $\varepsilon^{\mu\nu}$. Hence, $Sp(2) \simeq SU(2)$. The observation that $SU(2)$ can be viewed as $SO(-2)$ was first made by Penrose [132], who used it to compute $SU(2)$ invariants using "binors". His method does not generalize to $SO(n)$, for which spinors are needed to project onto totally antisymmetric reps (for the case $n = 2$, this is not necessary as there are no other reps). For $SU(2)$, *all* reps are fully symmetric (Young tableaux consist of a single row), and our 6-j's are all the 6-j's needed for computing $\underline{SU(2)} \simeq SO(3)$ group theoretic factors. More pedantically: $SU(2) \simeq$ spin $(3) \simeq \overline{SO(3)}$. Hence, all the Racah [135] and Wigner coefficients, familiar from the atomic physics textbooks. are special cases of our spinor/spinster 6-j's. Wigner's 3-j symbol [111]

$$\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M\end{pmatrix} \equiv \frac{(-1)^{j_1-j_2+M}}{\sqrt{2J+1}}(j_1 j_2 m_1 m_2|JM) \qquad (14.25)$$

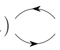
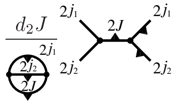
is really a Clebsch-Gordon coefficient with our 3-j as a normalization factor.

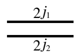
This may be expressed more simply in diagramatic form

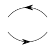
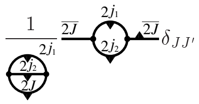
$$\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M\end{pmatrix} = $$  $$(14.26)$$

where we have not specified the phase convention on the righthand side as in the calculation of physical quantities such phases cancel. Factors of 2 appear because our integers $a, b, \ldots = 1, 2, \ldots$ count the numbers of $SU(2)$ 2-dimensional reps ($SO(3)$ spinors), while the usual $j_1, j_2, \ldots = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ labels correspond to $SO(3)$ angular momenta.

It is easy to verify (up to a sign) the completeness and orthogonality properties of Wigner's 3-j symbols

$$\sum_{J,M}(2J+1)$$  $$\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M\end{pmatrix}\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M\end{pmatrix} \sim \sum_J$$ 

$$=$$  $$\sim \delta_{m_1 m_1'}\delta_{m_2 m_2'} \qquad (14.27)$$

$$\sum_{m_1 m_2}$$  $$\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M\end{pmatrix}\begin{pmatrix} j_1 & j_2 & J' \\ m_1 & m_2 & M'\end{pmatrix} \sim$$  $$\delta_{JJ'}$$

$$= \frac{\delta_{JJ'}}{d_{2J}}\,\overline{2J} \sim \frac{\delta_{MM'}\delta_{JJ'}}{2J+1}\,. \qquad (14.28)$$

The expression (14.22) for our 3-j coefficient with $n = 2$ gives the expression

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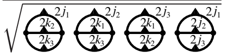
usually written as Δ in Racah's formula for $\left(\begin{smallmatrix} j & k & l \\ \alpha & \gamma & \gamma \end{smallmatrix}\right)$,

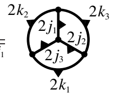
$$\frac{1}{\Delta(j,k,l)} = (-1)^{j+k+l}$$

$$= \frac{(j+k+l+1)!}{(j+k-l)!(k+l-j)!(l+j-k)!} . \qquad (14.29)$$

Wigner's 6-j coefficients are the same as ours, except that the 3-vertices are normalized as in (14.26)

$$\left\{\begin{smallmatrix} j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{smallmatrix}\right\} = \frac{1}{\sqrt{}}$$





$$, \qquad (14.30)$$

which gives Racah's formula using (14.23), with $n = 2$:

$$\left\{\begin{smallmatrix} j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{smallmatrix}\right\} = [\Delta(j_1 k_2 k_3)\Delta(k_1 j_2 k_3)\Delta(k_1 k_2 j_3)\Delta(j_1 j_2 j_3)]^{1/2}$$
$$\times \sum_t \frac{(-1)^t (t+1)!}{(t-j_1-j_2-j_3)!(t \leftrightarrow j_1-k_2-k_3)!(t-k_1-j_2-k_3)!(t-k_1-k_2-j_3)!(j_1+j_2-k_1+k_2-t)!(j_2}$$

14.3 HEISENBERG ALGEBRAS

The most interesting question raised by our labors is, what are spinsters? A sceptic would answer, that they are merely a trick for relating $SO(n)$ antisymmetric reps to $Sp(n)$ symmetric reps. That can be achieved without spinsters: indeed, Penrose [132] had observed many years ago, that $SO(-2)$ yields Racah coefficients in a much more elegant manner than the usual angular momentum manipulations. In chapter 13, we have also proved that for any scalar constructed from tensor invariants, $SO(-n) \simeq Sp(n)$. This theorem is based on elementary properties of permutations and establishes the equivalence between 6-j coefficients for $SO(-n)$ and $Sp(n)$, without reference to spinsters or any other Grassmann extensions.

Nevertheless, spinsters are the natural supersymmetric extension of spinors. They do not appear in the usual classifications, because they are infinite dimensional reps of $Sp(n)$. However, they are not as unfamiliar as they might seem; if we write the Grassmannian γ matrices for $Sp(2D)$ as $\gamma_\mu = (p_1, p_2, \ldots p_D, x_1, x_2 \ldots x_D)$ and choose $f_{\mu\nu}$ of form

$$f = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} , \qquad (14.32)$$

the defining commutator relation (14.2) is the defining relation for a Heisenberg algebra, except for a missing factor of i:

$$[p_i, x_j] = \delta_{ij}\mathbf{1} , \qquad i, j = 1, 2, \ldots D . \qquad (14.33)$$

It is well known that Heisenberg algebras have infinite dimensional reps, so the infinite dimensionality of spinsters is no surprise. If we include an extra factor of i into the definition of the "momenta" above, we find that spinsters resemble an antiunitary Grassmann-valued rep of the usual Heisenberg algebra. If there is any significance in these observations, it would be intriguing to consider relationship between superspace and the spinor/spinster reps of the orthosymplectic groups.

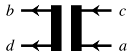
Chapter Fifteen

$SU(n)$ family of invariance groups

$SU(n)$ preserves the Levi-Civita tensor, in addition to the Kronecker δ of sect. 9.8. This additional invariant induces non-trivial decompositions of $U(n)$ reps. In this chapter, we show how the theory of $SU(2)$ reps (the quantum mechanics textbooks' theory of angular momentum) is developed by birdtracking; that $SU(3)$ is the unique group with the Kronecker delta and a rank-three antisymmetric primitive invariant; that $SU(4)$ is isomorphic to $SO(6)$; and that for $n \geq 4$, only $SU(n)$ has the Kronecker δ and rank-n antisymmetric tensor primitive invariants.

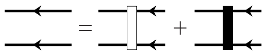
15.1 REPRESENTATIONS OF $SU(2)$

For $SU(2)$, we can construct an additional invariant matrix which would appear to induce a decomposition of $n \otimes \overline{n}$ reps

$$E_{b,\ d}^{a\ c} = \frac{1}{2}\varepsilon^{ac}\varepsilon_{bd} =$$  . (15.1)

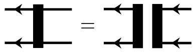
However, by (6.28) this can be written as a sum over Kronecker deltas and is not an independent invariant. So what does ε^{ac} do? It does two things; it removes the distinction between a particle and an antiparticle, (if q_a transforms as a particle, then $\varepsilon^{ab}q_b$ transforms as an antiparticle), and it reduces the reps of $SU(2)$ to the fully symmetric ones. Consider $n \otimes n$ decomposition (7.4)

$$\boxed{1}\otimes\boxed{2} = \boxed{1\,|\,2} + \bullet$$

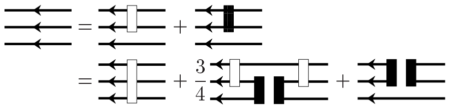
 (15.2)

$$2^2 = \frac{2\cdot 3}{2} + \frac{2\cdot 1}{2}\,.$$

The antisymmetric rep is a singlet

 . (15.3)

Now consider the $\otimes V^3$ and $\otimes V^4$ space decompositions, obtained by adding successive indices one at a time:



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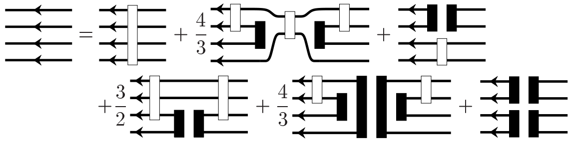
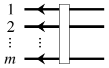
$$\boxed{1}\times\boxed{2}\times\boxed{3} = \boxed{1\,|\,2\,|\,3} + \boxed{1} + \boxed{3}$$



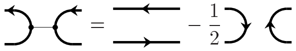
$$\boxed{1}\times\boxed{2}\times\boxed{3}\times\boxed{4} = \boxed{1\,|\,2\,|\,3\,|\,4} + \boxed{1\,|\,4} + \boxed{3\,|\,4} + \boxed{1\,|\,2} + \bullet + \bullet\,. \qquad (15.4)$$

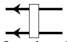
This is clearly leading us into the theory of $SO(3)$ angular momentum addition, described in any quantum mechanics textbook. We shall, anyway, persist a little while longer, just to illustrate how birdtracks can be used to recover some familiar results.

The projection operator for m-index rep is

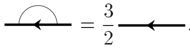
$$P_m =$$  . (15.5)

The dimension is $\mathrm{tr}\,P_m = \frac{2(2+1)(2+2)\dots(2+m-1)}{m!} = m+1$ (usually $m = 2j$, where j is the spin of the rep). The projection operator (7.10) for the adjoint rep (spin 1) is

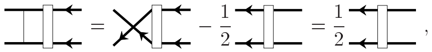
 . (15.6)

(This can be rewritten as  using (15.3)).

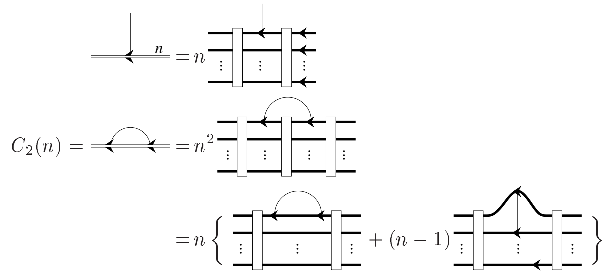
The quadratic casimir for the defining rep is

 . (15.7)

Using

 , (15.8)

we can compute the quadratic casimir for any rep

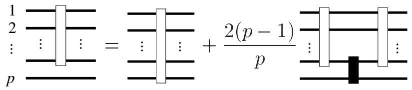


$$= n\left(\frac{3}{2} + \frac{n-1}{2}\right) \;\text{—}\; = \frac{n(n+2)}{2}\,. \qquad (15.9)$$

The Dynkin index for n-index rep is given by

$$\ell(n) = \frac{C_2(n)d_n}{C_2(2)d_2} = \frac{n(n+1)(n+2)}{24} . \tag{15.10}$$

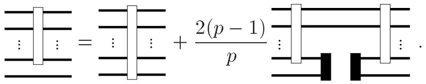
We can also construct Clebsch-Gordan coefficients for various Kronecker products. For example, $\lambda_p \otimes \lambda_1$ is given by



(15.11)

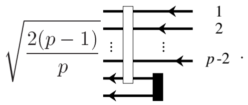
for any $U(n)$. For $SU(2)$ we have (15.3), so

$\boxed{1}\boxed{2}\boxed{\cdots}\boxed{p\text{-}1} \times \boxed{p} = \boxed{1}\boxed{2}\boxed{\cdots}\boxed{p} + \boxed{1}\boxed{2}\boxed{\cdots}\boxed{p\text{-}2}$



(15.12)

Hence, the Clebsch-Gordan for $\lambda_p \otimes \lambda_1 \to \lambda_{p-1}$ is



(15.13)

Actually, we have already given the complete theory of $SO(3)$ angular momentum in chapter 14, by giving explicit expressions for all Wigner 6-j coefficients (Racah coefficients), so we will not pursue this further here.

15.2 $SU(3)$ AS INVARIANCE GROUP OF A CUBIC INVARIANT

From experiments, we know that the hadrons are built from quarks and antiquarks, and that the hadron spectrum consists of

(i) mesons, each built from a quark and an antiquark;

(ii) baryons, each built from 3 quarks or antiquarks in a fully antisymmetric color combination;

(iii) no exotic states, *ie.* no hadrons built from other combinations of quarks and antiquarks.

We shall show here that for such hadronic spectrum, the color group can be only $SU(3)$.

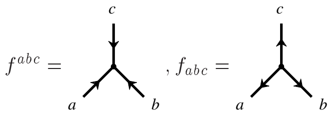
In the group theoretic language, the above three conditions are a list of the primitive invariants (color singlets) which defines the color group:

(i) One primitive invariant is δ_b^a, so the color group is a subgroup of $SU(n)$.

(ii) There is a cubic antisymmetric invariant f^{abc} and its dual f_{abc}.

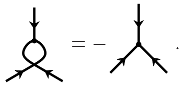
(iii) There are no further primitive invariants. We interpret this to mean that any invariant tensor can be written in terms of the tree contractions of δ_a^b, f^{abc} and f_{bca}.

In the birdtrack notation,



(15.14)

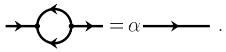
f_{abc} and f^{abc} are fully antisymmetric:



(15.15)

We can already see that the number is at least three, $n \geq 3$, as otherwise f_{abc} would be identically zero. Furthermore, f's must satisfy a normalization condition

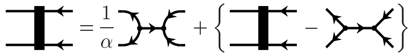
$$f^{abc}f_{bdc} = \alpha\delta_d^a$$



(15.16)

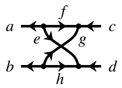
(For convenience we set $\alpha = 1$ in what follows.) If this were not true, eigenvalues of the invariant matrix $F_d^a = f^{abc}f_{bdc}$ could be used to split the n-dimensional rep in a direct sum of lower dimensional reps; but then n-dimensional rep would not be the defining rep (we would have several kinds of mesons).

$\otimes V^2$ *states.* According to (7.4), they split into symmetric and antisymmetric subspaces. The antisymmetric space is reduced to $n + \frac{n(n-3}{2}$ by the f^{abc} invariant:

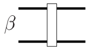


$$A_{ab,}{}^{cd} = \frac{1}{\alpha}f_{abe}f^{ecd} + \left\{A_{ab,}{}^{cd} - f_{abe}f^{ecd}\right\} . \tag{15.17}$$

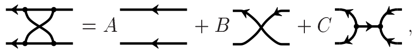
The symmetric subspace is not split by the $f_{abe}f^{ecd}$ invariant, which vanishes due to its antisymmetry. The simplest invariant matrix on the symmetric subspace involves four f's:

$K_{ab,}{}^{cd} =$  $= f_{aef}f_{bhg}f^{ceh}f^{dfg}$. (15.18)

As the symmetric subspace is not split, this invariant must have a single eigenvalue

$K_{ab,}{}^{cd} = \beta S_{ab,}{}^{cd} = \beta$  . (15.19)

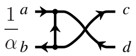
Tracing $K_{ab,}{}^{ad}$ fixes $\beta = \frac{2}{n+1}$. The assumption, that k is not an independent invariant, means that we do not allow the existence of exotic $qq\overline{qq}$ hadrons. The requirement, that all invariants be expressible as trees of contractions of the primitives



(15.20)

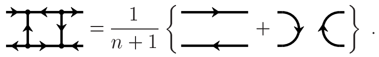
leads to the relation (15.19). The left-hand side is symmetric under index interchange $a \leftrightarrow b$, so $C = 0$ and $A = B$.

<u>*$V \otimes \overline{V}$ states*</u>. The simplest invariant matrix that we can construct from f's is

$G^{a\ d}_{b,c} = \frac{1}{\alpha}$  $= f^{aed} f_{bce}\,.$ (15.21)

By crossing (15.19), G satisfies a characteristic equation

$$G^2 = \frac{1}{n+1}\{\mathbf{1} + T\}$$

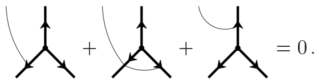


(15.22)

On the traceless subspace (7.8), this leads to

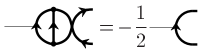
$$\left((G^2 - \frac{1}{n+1}\right) P_2 = 0\,, \qquad (15.23)$$

with eigenvalues $\pm 1/\sqrt{n+1}$. $V \otimes \overline{V}$ contains the adjoint rep, so at least one of the eigenvalues must correspond to the adjoint projection operator. We can compute the adjoint rep eigenvalue from the invariance condition (4.35) for f^{bcd}:



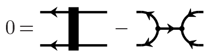
(15.24)

Contracting with f^{bcd}, we find



$$P_A G = -\frac{1}{2} P_A\,. \qquad (15.25)$$

Matching the eigenvalues, we obtain $\frac{1}{\sqrt{n+1}} = \frac{1}{2}$, so $n = 3$. Quarks can come in three colors only, and f_{abc} is proportional to the Levi-Civita tensor ε_{abc} of $SU(3)$. The invariant matrix G is not an independent invariant; the $n(n-3)/2$ dimensional antisymmetric space (15.17) has dimension zero, so G can be expressed in terms of Kronecker deltas:



$$0 = A_{ab,}{}^{cd} - G^{d\ c}_{a,b}\,. \qquad (15.26)$$

We have proven that the only group that satisfies the conditions (i) - (iii), at the beginning of this section, is $SU(3)$. Of course, it is well-known that the color group of physical hadrons is $SU(3)$, and this result might appear rather trivial. That it is not so will become clear from the further examples of invariance groups, such as the G_2 family of the next chapter.

15.3 LEVI-CIVITA TENSORS AND $SU(N)$

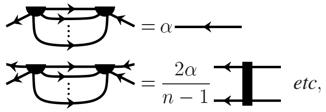
In chapter 12, we have shown that the invariance group for a skew-symmetric invariant f^{ab} is $Sp(n)$. In particular, for $f^{ab} = \varepsilon^{ab}$, the Levi-Civita tensor, the invariance group is $SU(2) = Sp(2)$. In the preceding section, we have proven that the invariance group of a skew-symmetric invariant f^{abc} is $SU(3)$, and that f^{abc} must be proportional to the Levi-Civita tensor. Now we shall show that for $f^{abc\ldots d}$ with r indices, the invariance group is $SU(r)$, and f is always proportional to the Levi-Civita tensor. $r = 2$ and $r = 3$ cases had to be treated separately, because it was possible to construct from f^{ab} and f^{abc} tree invariants on the $\overline{q} \times q \to \overline{q} \times q$ space which could reduce the group $SU(n)$ to a subgroup. For f^{ab}, $n \geq 4$ this is, indeed, what happens: $SU(n) \to Sp(n)$, for n even.

For $r \geq 4$, we assume here that the primitive invariants are δ_a^b and the fully skew-symmetric invariant tensors

$f^{a_1 a_2 \ldots a_r} =$ , $\quad ; \ r > 3\,.$

$f_{a_1 a_2 \ldots a_r} =$ . (15.27)

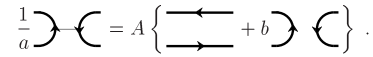
A fully antisymmetric object can be realized only in $n \geq r$ dimensions. By the primitiveness assumption



(15.28)

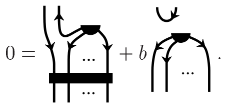
ie., various contractions of f's must be expressible in terms of δ's, otherwise there would exist additional primitives. (f invariants themselves have too many indices and cannot appear on the right hand side of the above equations.)

The projection operator for the adjoint rep can be built only from $\delta_b^a \delta_d^c$ and $\delta_d^a \delta_b^c$. From sect. 9.8, we know that this can give us only the $SU(n)$ projection operator (7.8), but just for fun we feign ignorance and write

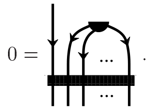


(15.29)

The invariance condition (6.58) on $f_{ab\ldots c}$ yields

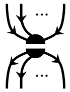
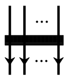


Contracting from the top, we get $0 = 1 + bn$. Antisymmetrizing all out legs, we get



(15.30)

Contracting with δ^a_b from the side, we get $0 = n - r$. As in (6.31), this defines the Levi-Civita tensor in n dimensions and can be rewritten as

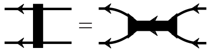
 $= n\alpha$  . (15.31)

(The conventional Levi-Civita normalization is $n\alpha = n!$). The solution $b = -\frac{1}{n}$ means that $T - i$ is traceless, *ie.*, the same as for the $SU(n)$ case considered in sect. 9.8. To summarize: the invariance condition forces $f_{abc\ldots c}$ to be proportional to the Levi-Civita tensor (because in n dimensions, a Levi-Civita tensor is the only fully antisymmetric tensor of rank n), and the primitives δ^a_b, $f_{ab\ldots d}$ (rank n) have $SU(n)$ as their unique invariance algebra.

15.4 $SU(4)$ - $SO(6)$ ISOMORPHISM

In the preceding sections, we have shown that if the primitive invariants are δ'^a_b, $f_{ab\ldots cd'}$, the corresponding Lie group is the defining rep of $SU(n)$, and $f_{ab\ldots cd}$ is proportional to the Levi-Civita tensor. However, there are still interesting things to be said about particular $SU(n)$'s. As an example, we will establish the $SU(4) \simeq SO(6)$ isomorphism.

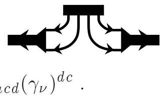
The antisymmetric $SU(4)$ rep is of dimension $d_A = \frac{4\cdot 3}{2} = 6$. Let us introduce clebsches



$$A_{ab,}{}^{cd} = \frac{1}{4}(\gamma^\mu)_{ab}(\gamma_\mu)^{cd}\,, \qquad \mu = 1, 2, \ldots, 6\,. \qquad (15.32)$$

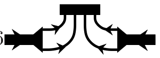
($\frac{1}{4}$ is a normalization chosen so that γ's will have the Dirac matrix normalization.)

The Levi-Civita tensor induces a quadratic symmetric invariant on the 6-dimensional space

$g_{\mu\nu} =$  $=$ 

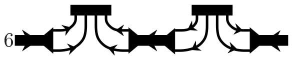
$$= \frac{1}{4}(\gamma_\mu)^{ab}\varepsilon_{bacd}(\gamma_\nu)^{dc}\,. \qquad (15.33)$$

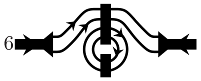
This invariant has an inverse

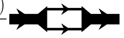
$g^{\mu\nu} =$  $= 6$  . (15.34)

The factor 6 is the normalization factor, fixed by the condition $g_{\mu\nu}g^{\nu\sigma} = \delta^\sigma_\mu$:

$g_{\mu\nu}g^{\nu\sigma} =$ 

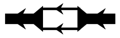
$= 6$ 

$= 6$ 

$= 6\frac{(n-3)}{4}\frac{(n-2)}{3}$ 

$=$  $= \delta^\sigma_\mu$. (15.35)

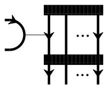
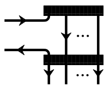
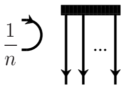
Here we have used (6.28), (15.32), and the orthonormality for clebsches

 $=$ 

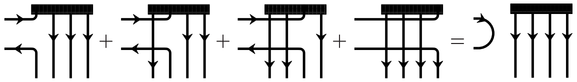
$$(\gamma_\mu)^{ab}(\gamma^\mu)_{ba} = 4\delta^\nu_\mu\,. \qquad (15.36)$$

As we have shown in chapter 10, the invariance group for a symmetric invariant $g_{\mu\nu}$ is $SO(d_A)$. One can check that the generators for the 6-dimensional rep of $SU(4)$, indeed, coincide with the defining rep generators of $SO(6)$, and that the dimension of the Lie algebra is in both cases 15.

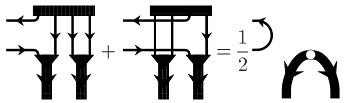
The invariance condition (6.58) for the Levi-Civita tensor is

$0 =$  $=$  $- \frac{1}{n}$  . (15.37)

For $SU(4)$ we have

 . (15.38)

Contracting with $(\gamma_\mu)^{ab}(\gamma_\nu)^{cd}$, we obtain



$$(\gamma_\mu)^{be}(\gamma_\nu)_{ab} + (\gamma_\mu)_{ad}(\gamma_\nu)^{de} = 2\delta^e_a g_{\mu\nu}\,. \qquad (15.39)$$

Here $(\gamma_\nu)_{ab} \equiv (\gamma_\nu)^{cd}\varepsilon_{dcab}$, and we recognize the Dirac equation (11.4). So the clebsches (15.32) are, indeed, the γ-matrices for $SO(6)$ (semi)spinor reps (11.65).

Chapter Sixteen

G_2 family of invariance groups

In this chapter, we begin the construction of all invariance groups which posses a symmetric quadratic and an antisymmetric cubic invariant in the defining rep. The resulting classification is summarized in fig. 16.1. We find that the cubic invariant must satisfy either the Jacobi relation (16.7) or the alternativity relation (16.11). In the former case, the invariance group can be any semi-simple Lie group in its adjoint rep; we pursue this possibility in the next chapter. The latter case is developed in this chapter; we find that the invariance group is either $SO(3)$ or the exceptional Lie group G_2. The problem of evaluation of $3n-j$ coefficients for G_2 is solved completely by the reduction identity (16.14). As a byproduct of the construction, we give a proof of the Hurwitz's theorem, sect. 16.6. We also demonstrate that the independent casimirs for G_2 are of order 2 and 6, by explicitly reducing the 4-th order casimir in sect. 16.5.

Consider the following list of primitive invariants:

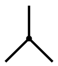
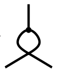
(i) δ^a_b, so the invariance group is a subgroup of $SU(n)$.

(ii) symmetric $g^{ab} = g^{ba}$, $g_{ab} = g_{ba}$, so the invariance group is a subgroup of $SO(n)$. We take this invariant in its diagonal, Kronecker delta form δ_{ab}.

(iii) a cubic antisymmetric invariant f_{abc}.

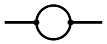
Primitiveness assumption requires that all other invariants can be expressed in terms of the tree contractions of δ_{ab} , f_{abc}.

In the diagrammatic notation, one keeps track of the antisymmetry of the cubic invariant by reading the indices off the vertex in a fixed order:

$f_{abc} =$  $= -$  $= -f_{acb}\,.$ (16.1)

The primitiveness assumption implies that the double contraction of a pair of f's is proportional to the Kronecker delta. We can use this relation to fix the overall normalization of f's:

$$f_{abc}f_{cbd} = \alpha\,\delta_{ad}$$

 $= \alpha$. (16.2)

For convenience, we shall often set $\alpha = 1$ in what follows.

The next step, in our construction, is to identify all invariant matrices on $\otimes V^2$ and construct the Clebsch-Gordan series for decomposition of 2-index tensors. There

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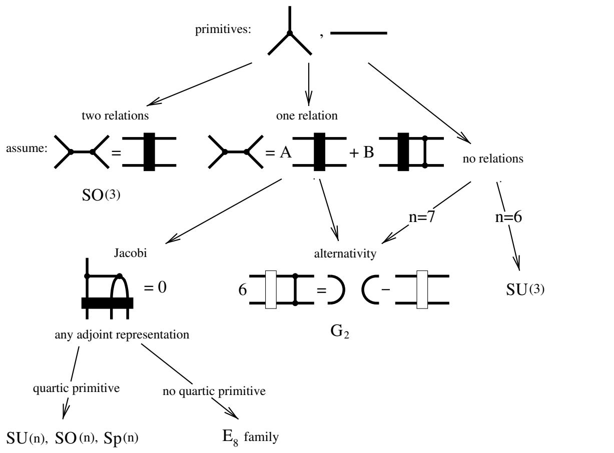


Figure 16.1 Logical organization of chapters 16–17. The invariance groups $SO(3)$ and G_2 are derived in this chapter, while the E_8 family is derived in chapter 17. PC: Bigger – G2 ?????.

are six such invariants: the three distinct permutations of indices of $\delta_{ab}\delta_{cd}$, and the three distinct permutations of free indices of $f_{abe}f_{ecd}$. For reasons of clarity, we shall break up the discussion into two steps. In the first step sect. 16.1, we assume that a linear relation between these six invariants exists. Pure symmetry considerations, together with the invariance condition, completely fix the algebra of invariants and restrict the dimension of the defining space, to either 3 or 7. In the second step sect. 16.3, we show that a relation *assumed* in the first step must exist because of the invariance condition.

Remark. Quarks and hadrons. An example of a theory, with above invariants, would be QCD with the hadronic spectrum consisting of following singlets:

(i) quark-antiquark mesons

(ii) mesons built of two quarks (or antiquarks) in a symmetric color combination

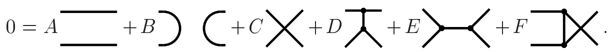
(iii) baryons built of three quarks (or antiquarks) in a fully antisymmetric color combination

(iv) no exotics, *ie.* no hadrons built from other combinations of quarks and antiquarks.

As we shall now demonstrate for this hadronic spectrum, the color group is either $SO(3)$, with quarks of three colors, or the exceptional Lie group G_2, with quarks of seven colors.

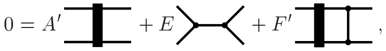
16.1 JACOBI RELATION

If the six invariant tensors mentioned above are not independent, they satisfy a relation of form



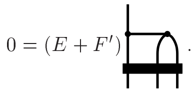
(16.3)

Antisymmetrizing a pair of indices yields



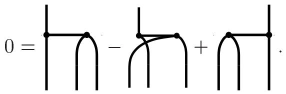
(16.4)

and antisymmetrizing any three indices yields



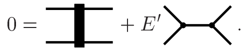
(16.5)

If the tensor itself vanishes, f's satisfy the *Jacobi relation* (4.47)



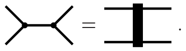
(16.6)

If $A' \neq 0$ in (16.4), the Jacobi relation relates the second and the third term



(16.7)

The normalization condition (16.2) fixes $E' = -1$.



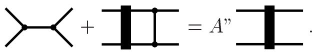
(16.8)

Contracting with δ_{ab}, we obtain $1 = (n-1)/2$, so $n = 3$. We conclude that if pair contraction of f's is expressible in terms of δ's, the invariance group is $SO(3)$, and f_{abc} is proportional to the 3-index Levi-Civita tensor. To spell it out; in 3 dimensions, an antisymmetric rank-3 tensor can take only one value, $f_{abc} = \pm f_{123}$ which can be set equal to ± 1 by appropriate normalization convention (16.2).

If $A' = 0$ in (16.4), the Jacobi relation is the only relation we have, and the adjoint rep of any simple Lie group is a possible solution. we return to this case in chapter 17.

16.2 ALTERNATIVITY AND REDUCTION OF F-CONTRACTIONS

If the Jacobi relation does not hold, we must have $E = -F'$ in (16.5) and (16.4) takes form



(16.9)

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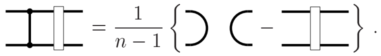
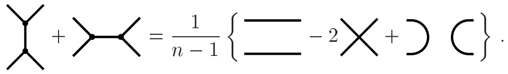
Contracting with δ_{ab} fixes $A" = 3/(n-1)$. Symmetrizing the top two lines and rotating the diagrams by 90^0, we obtain the *alternativity relation*:



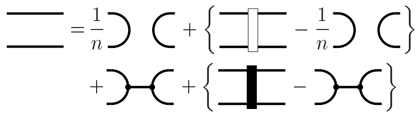
(16.10)

The name comes from the octonian interpretation given in sect. 16.5. Adding the two equations, we obtain

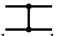


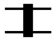
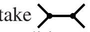
(16.11)

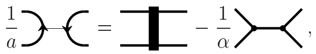
The Clebsch-Gordon decomposition of $\otimes V^2$ follows:



$$n^2 = 1 + \frac{(n-1)(n+2)}{2} + n + \frac{n(n-3)}{2}\,. \qquad (16.12)$$

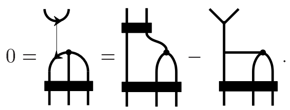
By (16.9), the invariant  is reducible on the antisymmetric subspace. By (16.10), it is also reducible on the symmetric subspace. The only independent $f \cdot f$ invariant is  which, by the normalization (16.2), is already the projection operator which projects the antisymmetric 2-index tensors onto the n-dimensional defining space. The dimensions of the reps are obtained by tracing the corresponding projection operators.

The adjoint rep,  of $SO(n)$, is now split into two reps. Which one is the new adjoint rep? That, we determine by considering (6.58), the invariance condition for f_{abc}. If we take  to be the projection operator for the adjoint rep, we again get the Jacobi condition with $SO(3)$ as the only solution. However, if we assume that the last term in (16.12) is the adjoint projection operator



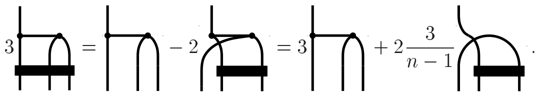
(16.13)

the invariance condition becomes a non-trivial condition:

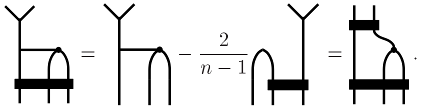


(16.14)

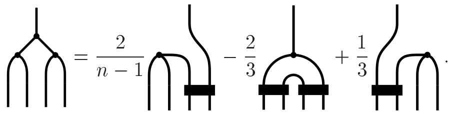
The last term can be simplified by (16.9) and (6.19)



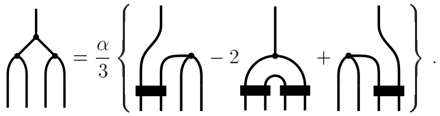
Substituting back into (16.14), yields



Expanding the last term and redrawing the equation slightly, we have



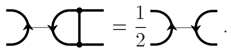
This equation is antisymmetric under interchange of the left and the right index pairs. Hence, $2/(n-1) = 1/3$, and the invariance condition is satisfied *only* for $n = 7$. Furthermore, the above relation gives us the G_2 *reduction identity*



(16.15)

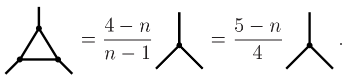
This identity is the key result of this chapter: it enables us to recursively reduce *all* contractions of products of δ-functions and pairwise contractions $f_{abc}f_{cde}$, and thus completely solves the problem of evaluating any casimir or $3n$-j coefficient of G_2.

The invariance condition (16.14) for f_{abc} implies that



(16.16)

The "triangle graph", for the defining rep, can be computed in two ways, either by contracting (16.10) with f_{abc}, or by contracting the invariance condition (16.14) with δ_{ab}:



(16.17)

So, the alternativity and the invariance conditions are consistent if $(n-3)(n-7) = 0$, *ie. only* for 3 or 7 dimensions. In the latter case, the invariance group is the exceptional Lie group G_2, and the above derivation is also a proof of Hurwitz's theorem, see sect. 16.5.

In this way, symmetry considerations together with the invariance conditions, suffice to determine the algebra satisfied by the cubic invariant. The invariance condition fixes the defining dimension to $n = 3$ or 7. Having assumed only that *a* cubic antisymmetric invariant exists, we find that if the cubic invariant is not a structure constant, it can be realized only in 7 dimensions, and its algebra is *completely* determined. The identity (16.14) plays the role analogous to that the Dirac relation $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbf{I}$ plays for evaluation of traces of products of Dirac gamma-matrices, described above in chapter 11. Just as the Dirac relation obviates

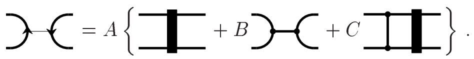
need for explicit reps of γ's, (16.14) reduces any $f \cdot f \cdot f$ contraction to a sum of terms linear in f and obviates any need for explicit construction of f's.

The above results now enable us to compute any group-theoretic weight for G_2 in two steps. First, we replace all adjoint rep lines by the projection operators P_A (16.13). The resulting expression contains Kronecker deltas and chains of contractions of f_{abc}, which can then be reduced by systematic application of the reduction identity (16.15).

16.3 PRIMITIVITY IMPLIES ALTERNATIVITY

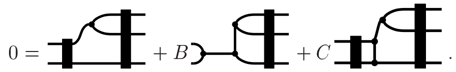
The only detail, which remains to be proven, is the assertion that the alternativity relation (16.10) follows from the primitiveness assumption. We complete the proof in this section. The proof is rather inelegant and can probably easily be streamlined.

If no relation (16.3) between the three $f \cdot f$ contraction is assumed, then by the primitiveness assumption the adjoint rep projection operator P_A is of the form

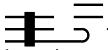


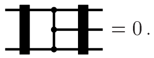
(16.18)

Assume that the Jacobi relation does not hold; otherwise this immediately reduces to $SO(3)$. The generators must be antisymmetric, as the group is a subgroup of $SO(n)$. Substitute the adjoint projection operator into the invariance condition (6.58) (or (16.14)) for f_{abc}:



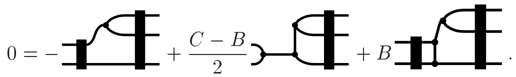
(16.19)

Resymmetrize this equation by contracting with . This is evaluated expanding with (6.19) and using a relation due to the antisymmetry of f_{abc}:



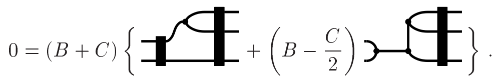
(16.20)

The result is:



(16.21)

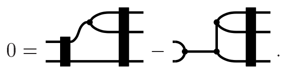
Multiplying (16.19) by B, (16.21) by C and subtracting, we obtain



(16.22)

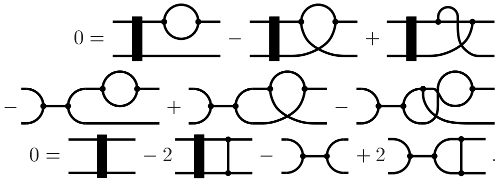
We return to the case $B + C = 0$ below, in (16.26).

If $B + C \neq 0$, by contracting with f_{abc} we get $B - C/2 = -1$, and

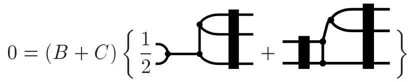


(16.23)

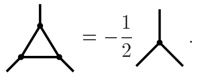
To prove that this is equivalent to the alternativity relation, we contract with , expand the 3-leg antisymmetrization, and obtain


(16.24)

The triangle subdiagram can be computed by adding (16.19) and (16.21)

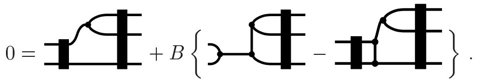


and contracting with . The result is

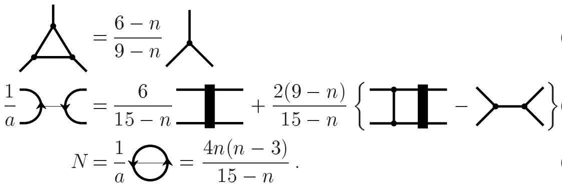

(16.25)

Substituting into (16.24), we recover the alternativity relation (16.10). Hence, we have proven that the primitivity assumption implies the alternativity relation for the case $B + C \neq 0$ in (16.22).

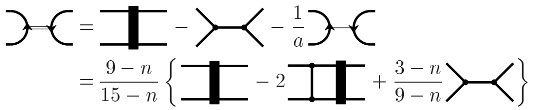
If $B + C = 0$, (16.19), becomes


(16.26)

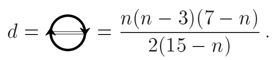
Using the normalization (7.37) and orthonormality conditions, we obtain


(16.27)
(16.28)
(16.29)

The remaining antisymmetric rep


(16.30)

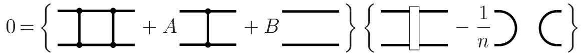
has dimension


(16.31)

The dimension cannot be negative, so $d \leq 7$. For $n = 7$, the projection operator (16.30) vanishes identically, and we recover the alternativity relation (16.10).

The Diophantine condition (16.31) has two further solutions: $n = 5$ and $n = 6$.

The $n = 5$ is eliminated by examining the decomposition of the traceless symmetric subspace in (16.12), induced by the invariant $\mathbf{Q} =$. By the primitiveness assumption, $\mathbf{Q}^2$ is reducible on the symmetric subspace



$$0 = (\mathbf{Q}^2 + A\mathbf{Q} + B)P_2\,.$$

Contracting the top two indices with δ_{ab} and $(T_i)_{ab}$, we obtain

$$\left(\mathbf{Q}^2 - \frac{1}{2}\frac{3-n}{9-n}\mathbf{Q} - \frac{5}{2}\frac{6-n}{(2+n)(9-n)}\mathbf{I}\right) P_2 = 0\,. \qquad (16.32)$$

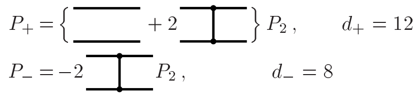
For $n = 5$, the roots of this equation are rational and the dimensions of the two reps, induced by decomposition with respect to $\mathbf{Q}$, are not integers. Hence, $n = 5$ is not a solution. We turn to the case $n = 6$ next.

16.4 SEXTONIANS

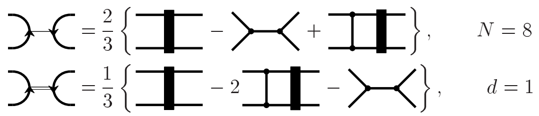
For the remaining $n = 6$ case the equation (16.30) reduces to

$$\left(\mathbf{Q} + \frac{1}{2}\right)\mathbf{Q}P_2 = 0 \qquad (16.33)$$

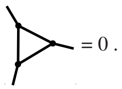
with the associated projection operators


(16.34)
(16.35)

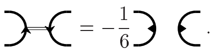
The adjoint (9.3) and the antisymmetric (9.42) projection operators are given by


(16.36)

Also



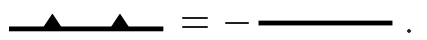
The existence of a 1-dimensional rep implies that $n = 6$ owns an associated skew-symmetric rank-2 invariant


(16.37)

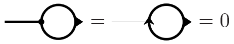
Here the normalization is chosen so that the warts

(16.38)

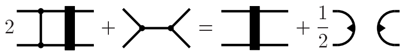
satisfy the fundamental wart identity (a wart is a symplectic invariant)


(16.39)

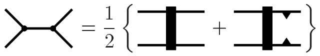
This invariant projects onto the 1-dimensional subspace (16.36) and is thus orthogonal to the defining and the adjoint reps


(16.40)

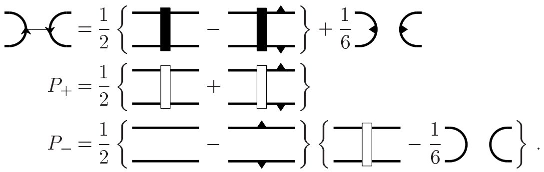
The cubic invariant can now be altogether eliminated in favor of the symplectic one: first we rewrite (16.36) as

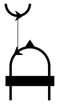


Antisymmetrizing the top two lines yields

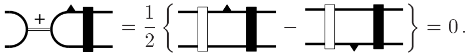

(16.41)

With this substitution the adjoint (16.36) and the two symmetric (16.35) rep projection operators are given by

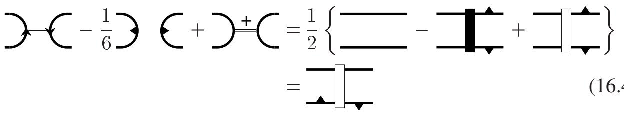

(16.42)

(The invariance condition 0=  is satisfied trivially).

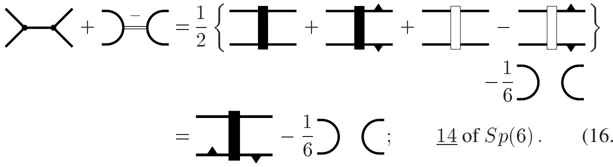
The 1-dimensional rep also satisfies the invariance condition, so it corresponds to a $U(1)$. Not only that, but P_+ also satisfies the invariance condition


(16.43)

Hence, the sum of the three adjoint reps


(16.44)

is the 8+1+12=21 dimensional adjoint rep of $Sp(6)$. The remaining reps also coalesce to $Sp(6)$ reps:



; $\underline{14}$ of $Sp(6)$. (16.45)

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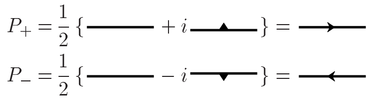
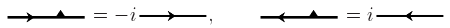
The fundamental wart identity (16.38) can be used to split the defining rep $6 \to 3 + \overline{3}$;



$$P_\pm P_\mp = 0 \qquad P_\pm P_\pm = P_\pm \tag{16.46}$$

The wart has eigenvalues $\pm$:

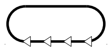

(16.47)

The significance of this 6-dimensional alternative algebra, intermediate between the complex quaternions and octonions, named sextonians by Westbury [203], was first appreciated by J.M. Landsberg, L. Manivel and B.W. Westbury [188].

16.5 CASIMIRS FOR G_2

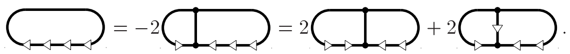
In this section, we prove that the independent casimirs for G_2 are of order 2 and 6, as indicated in the table 7.1. As G_2 is a subgroup of $SO(7)$, its generators are antisymmetric, and only even order casmirs are nonvanishing.

The quartic casimir, (in the notation of (7.9))

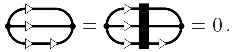


$$= \mathrm{tr}\, X^4 = \sum_{ijkl} x_i x_j x_k x_l \mathrm{tr}\, (T_i T_j T_k T_l) \,,$$

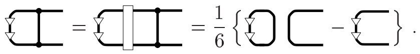
can be reduced by manipulating it with the invariance condition (6.58)



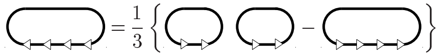
The last term vanishes by further manipulation with the invariance condition


(16.48)

The remaining term is reduced by the alternativity relation (16.10)



This yields the explicit expression for the reduction of quartic casimirs in the defining rep of G_2



$$\mathrm{tr}\, X^4 = \frac{1}{4}\left(\mathrm{tr}\, X^2\right)^2 \,. \tag{16.49}$$

As the defining rep is 7-dimensional, the characteristic equation (7.10) reduces the 8th and all higher order casimirs. Hence, the independent casimirs for G_2 are of order 2 and 6.

16.6 HURWITZ'S THEOREM

Definition (Curtis [34]): A normed algebra A is an $n+1$ dimensional vector space over a field F with a product xy such that

$$\begin{aligned} (i)\ x(cy) &= (cx)y = c(xy)\,, \qquad c \in F \\ (ii)\ x(y+z) &= xy + xz\,, \qquad x, y, z \in A \\ (x+y)z &= xz + yz, \end{aligned}$$

and a non-degenerate quadratic norm which permits composition

$$(iii) \qquad N(xy) = N(x)N(y)\,, \qquad N(x) \in F. \tag{16.50}$$

Here F will be the field of real numbers. Let $\{\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis of A over F:

$$x = x_0\mathbf{e}_0 + x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n\,, \qquad x_a \in F\,, \quad \mathbf{e}_a \in A\,. \tag{16.51}$$

It is always possible to choose $\mathbf{e}_o = \mathbf{I}$ (see Curtis [34]). The product of remaining bases must close the algebra

$$\mathbf{e}_a\mathbf{e}_b = -d_{ab}\mathbf{I} + f_{abc}\mathbf{e}_c\,, \qquad d_{ab}, f_{abc} \in F \qquad a, \ldots, c = 1, 2, \ldots, n\,. \tag{16.52}$$

The norm in this basis is

$$N(x) = x_0^2 + d_{ab}x_a x_b. \tag{16.53}$$

From the symmetry of the associated inner product (Tits [151])

$$(x, y) = (y, x) = -\frac{N(x+y) - N(x) - N(y)}{2}, \tag{16.54}$$

it follows that $-d_{ab} = (\mathbf{e}_a, \mathbf{e}_b) = (\mathbf{e}_b, \mathbf{e}_a)$ is symmetric, and it is always possible to choose bases $\mathbf{e}_a$ such that

$$\mathbf{e}_a\mathbf{e}_b = -\delta_{ab} + f_{abc}\mathbf{e}_c. \tag{16.55}$$

Furthermore, from

$$\begin{aligned} -(xy, x) &= \frac{N(xy+x) - N(x)N(y)}{2} = N(x)\frac{N(y+1) - N(y) - 1}{2} \\ &= N(x)(y, 1), \end{aligned} \tag{16.56}$$

it follows that $f_{abc} = (\mathbf{e}_a, \mathbf{e}_b, \mathbf{e}_c)$ is fully antisymmetric. [In Tits' notation [151], the multiplication tensor f_{abc} is replaced by a cubic antisymmetric form (a, a', a''), his equation (14)]. The composition requirement (16.50) expressed in terms of bases (16.51) is

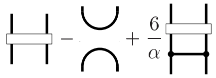
$$\begin{aligned} 0 &= N(xy) - N(x)N(y) \\ &= x_a x_b y_c y_d\,(\delta_{ac}\delta_{bd} - \delta_{ab}\delta_{cd} + f_{ace}f_{cbd})\,. \end{aligned} \tag{16.57}$$

To make a contact with sect. 16.2, we introduce diagrammatic notation (factor $i\sqrt{6/\alpha}$ adjusts the normalization to (16.2))

$$f_{abc} = i\sqrt{\frac{6}{\alpha}}\ \ . \tag{16.58}$$

Diagrammatically, (16.57) is given by

$$0 = \quad - \quad + \frac{6}{\alpha} \quad . \tag{16.59}$$



This is precisely the relation (16.10) which we have proven to be nontrivially realizable only in 3 and 7 dimensions. The trivial realizations are $n = 0$ and $n = 1$, $f_{abc} = 0$. So we have inadvertently proven

Hurwitz's theorem (Curtis [34]): *$n+1$ dimensional normed algebras over reals exist only for $n = 0, 1, 3, 7$ (real, complex, quaternion, octonion).*

We call (16.10) the *alternativity* relation, because it can also be obtained by substituting (16.55) into the alternativity condition for octonions [143]

$$\begin{aligned} [xyz] &\equiv (xy)z - x(yz)\,, \\ [xyz] &= [zxy] = [yzx] = -[yxz]\,. \end{aligned} \tag{16.60}$$

Cartan [20] was first to note that $G_2(7)$ is the isomorphism group of octonions, *ie.* the group of transformations of octonion bases (written here in the infinitesimal form)

$$\mathbf{e}'_a = (\delta_{ab} + iD_{ab})\mathbf{e}_b$$

which preserve the octonionic multiplication rule (16.55). The reduction identity (16.15) was first derived by Behrends *et al*. [9] [in very different notation, their equation (16)]. Tits also constructed the adjoint rep projection operator for $G_2(7)$ by defining the derivation on an octonion algebra as

$$Dz = <x, y> z = -\frac{1}{2}((x \cdot y) \cdot z) + \frac{3}{2}[(y, z)x - (x, z)y],$$

[Tits 1966, equation (23)] where

$$\mathbf{e}_a \cdot \mathbf{e}_b \equiv f_{abc}\mathbf{e}_c, \tag{16.61}$$

$$(\mathbf{e}_a, \mathbf{e}_b) \equiv -\delta_{ab}. \tag{16.62}$$

Substituting $x = x_a\mathbf{e}_a$, we find

$$(Dz)_d = -3x_a y_b \left(\frac{1}{2}\delta_{ab}\delta_{bd} + \frac{1}{6}f_{abe}f_{ecd}\right) z_c\,. \tag{16.63}$$

The term in the brackets is just the $G_2(7)$ adjoint rep projection operator P_A in (16.13), with normalization $\alpha = -3$.

16.7 REPRESENTATIONS OF G_2

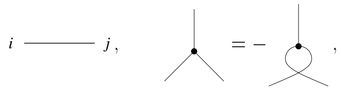
G_2 is characterized by the fully antisymmetric cubic primitive invariant f_{abc}. Contracting with f_{abc}, we are able to reduce any column higher than two boxes. Hence, reps of G_2 are specified by Young tableaux of form $(qp00\ldots)$. Patera and Sankoff [128] have chosen to label the simple roots in such a way that the correspondence is

Chapter Seventeen

E_8 family of invariance groups

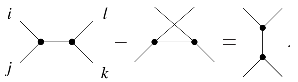
In this chapter we continue the construction of invariance groups characterized by a symmetric quadratic and an antisymmetric cubic primitive invariant. In the preceding chapter we proved that the cubic invariant must either satisfy the alternativity relation (16.11), or the Jacobi relation (16.7), and showed that the first case has $SO(3)$ and G_2 as the only interesting solutions.

Here we pursue the second possibility, and determine all invariance groups which preserve a symmetric quadratic (4.26) and an antisymmetric cubic primitive invariant (4.44),



(17.1)

with the cubic invariant satisfying the Jacobi relation (4.46)

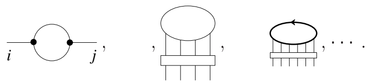


(17.2)

Our task is twofold: we need to

(i) enumerate all Lie algebras defined by the primitives (17.1)-(17.1). The key idea here is the primitiveness assumption (3.36). By requiring that the list of (17.1) is the full list of primitive invariants, *ie.* that any invariant tensor can be expressed as a linear sum over the tree invariants constructed from the quadratic and the cubic invariants, we are classifying those invariance groups for which *no quartic primitive* invariant exists in the adjoint rep (see fig. 16.1).

(ii) demonstrate that we can compute all $3n$-j coefficients (or casimirs, or vacuum bubbles); the ones up to 12-j are listed in table 5.1. Due to the antisymmetry (17.1) of structure constants and the Jacobi relation (17.1), we need to concentrate on evaluation of only the the symmetric casimirs, a subset of (7.13):



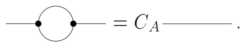
(17.3)

Here cheating a bit and peeking into the list of the Betti numbers table 7.1 offers some moral guidance.

We accomplish here most of (i): the Diophantine conditions (17.14)-(17.19) yield all of the $E_8 ting$ family Lie algebras, and no stragglers, but we fail to prove that there exist no further Diophantine conditions, and that all of these groups actually exist. We are much further from demonstrating (ii): The projection operators for E_8 family, given in tables 17.1 and 17.2 enable us to evaluate diagrams with internal loops of length 5 or smaller, so we have no proof that *any* vacuum bubble can be so evaluated.

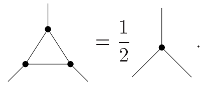
As, by assumption, the defining rep satisfies the Jacobi relation (17.2), the defining rep is in this case also A, the adjoint rep of some Lie group. Hence, in this chapter we denote the dimension of the defining rep by N, the cubic invariant by the Lie algebra structure constants $-iC_{ijk}$, and draw the invariants with the thin (adjoint) lines, as in (17.1) and (17.2).

The assumption that the defining rep is irreducible means in this case that the Lie group is simple, and the quadratic casimir (Cartan-Killing tensor) is proportional to the identity



(17.4)

In this chapter we shall usually choose normalization $C_A = 1$. The Jacobi relation (17.2) reduces a loop with three structure constants

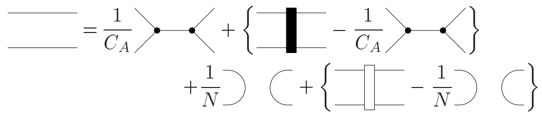


(17.5)

Remember the graph (1.1)? The one graph that launched this whole odyssey? In order to learn how to reduce such loops with four structure constants we turn to the reduction of the $A \otimes A$ space.

17.1 TWO-INDEX TENSORS

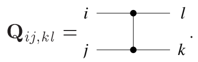
By the reasoning of sects. 10.1–10.3 existence of the quadratic and cubic invariants (17.1) induces a decomposition of $A \otimes A$ tensors into four subspaces:



(17.6)

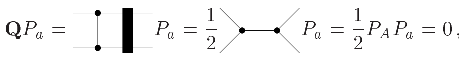
$$\mathbf{1} = P_\square + P_{\text{日}} + P_\bullet + P_s \,.$$

Consider $A \otimes A \to A \otimes A$ invariant matrix



(17.7)

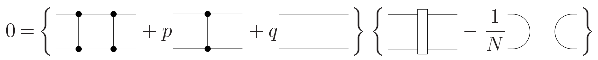
By the Jacobi relation (17.2), $\mathbf{Q}$ has zero eigenvalue on the antisymmetric subspace



(17.8)

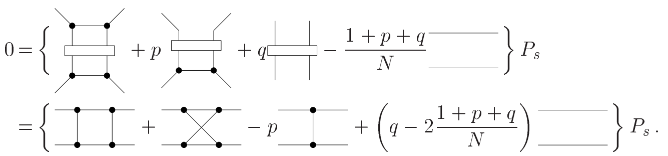
so $\mathbf{Q}$ can decompose only the symmetric subspace.

By the primitiveness assumption, the 4-index loop invariant $\mathbf{Q}^2$ is not an independent invariant, but is expressible in terms of any full linearly independent set of the 4-index tree invariants $\mathbf{Q}_{ij,k\ell}$, $f_{ijm}f_{mk\ell}$ and δ_{ij}'s constructed from the primitive invariants (17.1). The assumption that there exists no primitive quartic invariant is the *defining relation* for the E_8 family. On the traceless symmetric subspace, this implies that $\mathbf{Q}^2 P_s$ satisfies a relationship of form

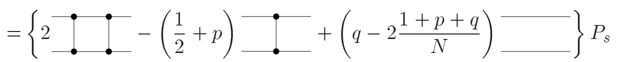


$$0 = (\mathbf{Q}^2 + p\mathbf{Q} + q)P_s\,. \qquad (17.9)$$

The coefficients p, q now follow from symmetry considerations and the Jacobi relation. Rotate each term in the above equation by 90^o and the project onto the traceless symmetric subspace;



Jacobi relation (4.47) relates the second term to the first:



$$0 = \left(\mathbf{Q}^2 - \frac{1+2p}{4}\mathbf{Q} + \frac{q}{2} - \frac{1+p+q}{N}\right) P_s\,. \qquad (17.10)$$

Comparing the coefficients in (17.9) and (17.10), we obtain the characteristic equation for $\mathbf{Q}$

$$\left(\mathbf{Q}^2 - \frac{1}{6}\mathbf{Q} - \frac{5}{3(N+2)}\right) P_s = (\mathbf{Q}-\lambda)(\mathbf{Q}-\lambda^*)P_s = 0\,. \qquad (17.11)$$

We shall use this equation to obtain a Diophantine condition on admissible dimensions of the adjoint rep. Either eigenvalue of $\mathbf{Q}$ satisfies the characteristic equation

$$\lambda^2 - \frac{1}{6}\lambda - \frac{5}{3(N+2)} = 0\,,$$

hence, N can be expressed in terms of the eigenvalue

$$N+2 = \frac{5}{3\lambda(\lambda - 1/6)} = 10\left\{(6-\lambda^{-1}) - 12 + \frac{6^2}{6-\lambda^{-1}}\right\}\,. \qquad (17.12)$$

It is convenient to reparametrize the two eigenvalues as

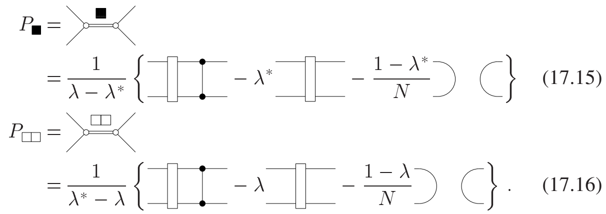
$$\lambda = -\frac{1}{m-6}\,, \qquad \lambda^* = \frac{1}{6}\frac{m}{m-6}\,. \qquad (17.13)$$

In terms of the parameter m, the dimension of the adjoint representation is given by

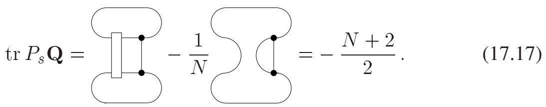
$$N = -122 + 10m + \frac{360}{m}\,. \qquad (17.14)$$

As N is an integer, allowed m are rationals $m = P/Q$ built from Q any combination of subfactors of denominator $60 = 2^2\cdot 3\cdot 5$ and numerator $P = 0, 1, 2,$ or 5. P and Q are relative primes, and there are 45 distinct allowed rationals in all. As either root λ, λ^* solves (17.14), the solutions are symmetric under interchange $m/6 \leftrightarrow 6/m$, so we need to check only the 27 rationals $m > 6$. We postpone the Diophantine analysis to sect. 17.4.

The associated projection operators (3.45) are



To compute the dimensions of the two subspaces we first evaluate



The dimension of $\square\square$ is then given by

$$d_{\square\square} = \mathrm{tr}\,P_{\square\square} = \frac{(N+2)(1/\lambda + N - 1)}{2(1-\lambda^*/\lambda)}\,, \qquad (17.18)$$

and $d_{\blacksquare}$ is obtained by interchanging λ and λ^*. Substituting (17.14), (17.13) leads to

$$d_{\square\square} = \frac{5(m-6)^2(5m-36)(2m-9)}{m(m+6)}$$
$$d_{\blacksquare} = \frac{270(m-6)^2(m-5)(m-8)}{m^2(m+6)}\,. \qquad (17.19)$$

The solutions that survive the Diophantine conditions form the E_8 *family*, listed in table 17.1.

To summarize, in absence of a primitive 4-index invariant, $A \otimes A$ decomposes into 5 irreducible reps

$$\mathbf{1} = P_{\square} + P_{\substack{\square\\\square}} + P_{\bullet} + P_{\blacksquare} + P_{\square\square} \qquad (17.20)$$

The decomposition is parametrized by integer m and is possible only if N and $d_{\square\square}$ satisfy Diophantine conditions (17.14), (17.19).

Perhaps this is not apparent, but what we have accomplished is a reduction of the adjoint rep 4-vertex box(....) for, as will turn out, all exceptional Lie groups!

The general strategy for decomposition of higher tensor products is as follows; the equation (17.10) reduces $\mathbf{Q}^2$ to $\mathbf{Q}$, P_r weighted by the eigenvalues λ, λ^*. For higher tensor products, we shall use the same result to decompose symmetric subspaces.

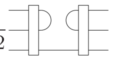
Dynkin labels	$\square \otimes \square$	=	$\square$	+	$\square\square$	+	$\bullet$	+	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	+	$\blacksquare$
E_8	$(10000000)\otimes(10000000)$	=	(10000000)	+	(01000000)	+	(00000000)	+	(20000000)	+	(00000010)
E_7	$(1000000)\otimes(1000000)$	=	(1000000)	+	(0100000)	+	(0000000)	+	(2000000)	+	(0000100)
E_6	$(000001)\otimes(000001)$	=	(000001)	+	(001000)	+	(000000)	+	(000002)	+	(100010)
F_4	$(1000)\otimes(1000)$	=	(1000)	+	(0100)	+	(0000)	+	(2000)	+	(0010)
D_4	$(0100)\otimes(0100)$	=	(0100)	+	(1010)	+	(0000)	+	(0200)	+	$(2000)+(0001)$
G_2	$(10)\otimes(10)$	=	(10)	+	(03)	+	(00)	+	(20)	+	(02)
A_2	$(11)\otimes(11)$	=	(11)	+	$(12)+(21)$	+	(00)	+	(22)	+	(11)
A_1	$(2)\otimes(2)$	=	(2)	+	(0)	+	(4)	+	(4)		
Dimensions	N^2	=	N	+	$\frac{N(N-3)}{2}$	+	1	+	d_-	+	d_+
E_8	248^2	=	248	+	$30,380$	+	1	+	$27,000$	+	$3,875$
E_7	133^2	=	133	+	$8,645$	+	1	+	$7,371$	+	$1,539$
E_6	78^2	=	78	+	$2,925$	+	1	+	$2,430$	+	650
F_4	52^2	=	52	+	$1,274$	+	1	+	$1,053$	+	324
D_4	28^2	=	28	+	350	+	1	+	300	+	$35+35+35$
G_2	14^2	=	14	+	77	+	1	+	77	+	27
A_2	8^2	=	8	+	$10+\overline{10}$	+	1	+	27	+	8
A_1	3^2	=	3	+	0	+	1	+	5	+	0

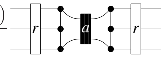
Table 17.1 E_8 family Clebsch-Gordon series for $\otimes A^2$. The corresponding projection operators are listed in (17.6), (17.15) and (17.16). The parameter m is defined in (17.13).

We shall refer to a decomposition as "boring" if it brings no new Diophantine condition. As $\mathbf{Q}$ acts only on the symmetric subspaces, decompositions of antisymmetric subspaces will are always boring, as was already the case in (17.8). We illustrate the technique by working out the decomposition of $\mathrm{Sym}^3 A$ and $V \otimes V$ in the next two sections.

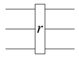
17.2 DECOMPOSITION OF SYM$^3 A$

Consider $\mathrm{Sym}^3 A$, the fully symmetrized subspace of $\otimes^3 A$. As the first step, project out the A and $A \otimes A$ content of $\mathrm{Sym}^3 A$:

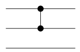
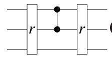
$$P_\square = \frac{3}{N+2} \qquad (17.21)$$


$$P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \frac{6(N+1)(N^2-4)}{5(N^2+2N-5)} \qquad (17.22)$$


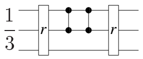
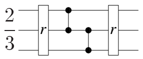
$P_\square$ projects out $\mathrm{Sym}^3 A \to A$, and $P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ projects out the antisymmetric subspace (17.6) $\mathrm{Sym}^3 A \to \wedge^2 A$. The ugly prefactor is a normalization, and will play no role in what follows. We shall decompose the remainder of the $\mathrm{Sym}^3 A$ space

$$P_r = S - P_\square - P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \qquad (17.23)$$


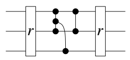
by the invariant tensor $\mathbf{Q}$ restricted to the P_r remainder subspace

$$\mathbf{Q} = \qquad , \qquad \hat{\mathbf{Q}} = \qquad \hat{\mathbf{Q}} = P_r \mathbf{Q} P_r \qquad (17.24)$$



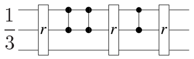
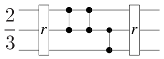
We can partially reduce $\hat{\mathbf{Q}}^2$ using (17.11) but symmetrization leads also to a new invariant tensor

$$\hat{\mathbf{Q}}^2 = \frac{1}{3} \qquad + \frac{2}{3} \qquad . \qquad (17.25)$$



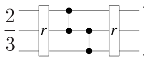
A calculation that requires applications of the Jacobi relation (4.47), symmetry identities such as

$$= 0\,, \qquad (17.26)$$


and relies on the fact that P_r contains no $A, \otimes V A$ subspaces yields

$$\hat{\mathbf{Q}}^3 = \frac{1}{3} \qquad + \frac{2}{3} \qquad . \qquad (17.27)$$



Reducing by (17.11) leads to

$$\hat{\mathbf{Q}}^3 = (\lambda + \lambda^*) \left\{ \frac{1}{3}\hat{\mathbf{Q}}^2 + \frac{2}{3} \qquad \right\} - \lambda\lambda^* \hat{\mathbf{Q}} \qquad (17.28)$$


The extra tensor can be eliminated by (17.25), and the result is a cubic equation for $\hat{\mathbf{Q}}$ (where we have substituted $\lambda + \lambda^* = 1/6$, using (17.11)):

$$0 = \left(\hat{\mathbf{Q}} - 1/18\right)\left(\hat{\mathbf{Q}} - \lambda\right)\left(\hat{\mathbf{Q}} - \lambda^*\right) P_r\,. \qquad (17.29)$$

The projection operators for the corresponding three subspaces are given by (3.45)

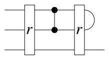
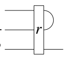
$$P_3 = \frac{\left(\hat{\mathbf{Q}} - \lambda\right)\left(\hat{\mathbf{Q}} - \lambda^*\right)}{(1/18 - \lambda)(1/18 - \lambda^*)} P_r \tag{17.30}$$
$$= -\frac{162\,(m-6)^2}{(m+3)(m+12)} \left\{ \hat{\mathbf{Q}}^2 - \frac{1}{6}\hat{\mathbf{Q}} - \frac{36}{6m(\frac{2}{m}-6)^2} \right\} P_r$$

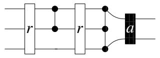
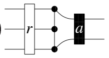
$$P_4 = \frac{\left(\hat{\mathbf{Q}} - 1/18\right)\left(\hat{\mathbf{Q}} - \lambda^*\right)}{(\lambda - 1/18)(\lambda - \lambda^*)} P_r \tag{17.31}$$
$$= \frac{54\,(m-6)^2}{(m+3)(m+6)} \left\{ \hat{\mathbf{Q}}^2 - \frac{m-24}{18(m-6)}\hat{\mathbf{Q}} + \frac{1}{18(m-6)} \right\} P_r$$

$$P_{4^*} = \frac{\left(\hat{\mathbf{Q}} - 1/18\right)\left(\hat{\mathbf{Q}} - \lambda\right)}{(\lambda^* - 1/18)(\lambda^* - \lambda)} P_r \tag{17.32}$$
$$= \frac{108\,(m-6)^2}{(m+6)(m+12)} \left\{ \hat{\mathbf{Q}}^2 - \frac{2(m-3)}{9(m-6)}\hat{\mathbf{Q}} + \frac{m}{108(m-6)} \right\} P_r$$

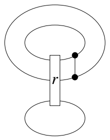
The presumption is (still to be proved for a general tensor product) that only reductions occur in the symmetric subspaces, always via the $\mathbf{Q}$ characteristic equation (17.11). As the overall scale of $\mathbf{Q}$ is arbitrary, there is only one rational parameter in the problem, either λ/λ^* or m, or whatever seems convenient. Hence, all dimensions and any coefficients will be ratios of polynomials in m.

To proceed, we follow the method outlined in appendix A. On $P_\square$, $P_{\boxminus}$ subspaces $S\mathbf{Q}$ has eigenvalues

$S\mathbf{Q}P_\square =$  $= \frac{1}{3}$  $\to \lambda_\square = 1/3$ (17.33)

$S\mathbf{Q}P_{\boxminus} =$  $= (\lambda + \lambda^*)$  $\to \lambda_{\boxminus} = 1/6$ (17.34)

so the eigenvalues are $\lambda_\square = 1/3$, $\lambda_{\boxminus} = 1/6$, $\lambda_3 = 1/18$, $\lambda_4 = \lambda$, $\lambda_{4^*} = \lambda^*$. The dimension formulas (A.8) require evaluation of

$\mathrm{tr}\, S\mathbf{Q} =$  $= -\frac{N(N+2)}{6}$ (17.35)

$$\mathrm{tr}\,(S\mathbf{Q})^2 = \texttt{birdTrack} = \frac{N(3N+16)}{36}\,. \tag{17.36}$$

Substituting into (A.8) we obtain the dimensions of the three new reps:

$$d_3 = \frac{27(m-5)(m-8)(2\,m-15)(2\,m-9)(5\,m-36)(5\,m-24)}{m^2(3+m)(12+m)}$$

$$d_4 = \frac{10(m-6)^2(m-5)(m-1)(2\,m-9)(5\,m-36)(5\,m-24)}{3\,m^2(6+m)(12+m)}$$

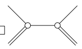
$$d_{4^*} = \frac{5(m-5)(m-8)(m-6)^2(2\,m-15)(5\,m-36)}{m^3(3+m)(6+m)}(36-m) \tag{17.37}$$

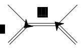
The integer solutions of the above Diophantine conditions are listed in table 17.2.

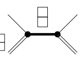
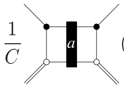
The main result of all this heavy birdtracking is that $N > 248$ is excluded by the positivity of d_{4^*}, and $N = 248$ is special, as $P_{4^*} = 0$ implies existence of a tensorial identity on the $\mathrm{Sym}^3 A$ subspace. That dimensions should all factor into terms linear in m is altogether not obvious at this point.

17.3 DECOMPOSITION OF $\square \otimes \blacksquare$

The decomposition of $\otimes^2 A$ tensors has split the traceless symmetric subspace into a pair of reps which we denoted by $\square\!\square$, $\blacksquare$. Now we turn to the decomposition of $\square \otimes \blacksquare$ Kronecker product. We commence by identifying the A and $\otimes A^2$ content of the $\square \otimes \blacksquare \in \otimes A^3$ Kronecker product. The $\square$, $\blacksquare$ and $\boxminus$ components of $\square \otimes \blacksquare$ are projected out by

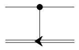
$P_\square = K_\square$  (17.38)

$P_\blacksquare = K_\blacksquare$  (17.39)

$P_{\boxminus} = K_{\boxminus}$  $= \frac{1}{C}$  $(1 - P_\square)\,,$ (17.40)

where the $\square \otimes \blacksquare$ vertex is given by (17.15), and $\boxminus$ is the not-adjoint antisymmetric rep in (17.6). In this section double line denotes $\blacksquare$ rep, and K_α are normalization factors given by ratios of dimensions and appropriate Dynkin indices (5.7) (or 3-j coefficients). As we shall not need them here, we do not write them out explicitly.

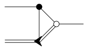
We shall use the invariant tensor

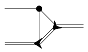
$\mathbf{R} =$  $= 2\,,$ (17.41)

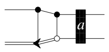
the restriction of the $\mathbf{Q}$ from (**??**) to the $\blacksquare \otimes \square$ space, to decompose the remainder subspace

$$P_r = 1 - P_\square - P_\blacksquare - P_{\boxminus}\,. \tag{17.42}$$

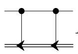
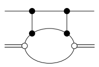
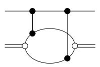
The eigenvalue of $\mathbf{R}$ on each of the above subspaces follows from invariance conditions (**??**) and the eigenvalue equation (3.50) $QP_\blacksquare = \lambda P_\blacksquare$ see also (**??**):

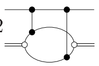
$\mathbf{R}P_\square =$  $= (1-\lambda)P_\square$ (17.43)

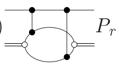
$\mathbf{R}P_\blacksquare =$  $= \frac{1}{2}P_\blacksquare$ (17.44)

$RP_{\boxminus} =$  $= \left(\frac{1}{2} - \lambda\right) P_{\boxminus}\,.$ (17.45)

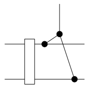
The characteristic equation for R projected to the remainder subspace (cf. (3.54)) is obtained by evaluating R^2 and R^3:

$R^2 P_r =$  $P_r = 2 \{$  $+$  $\} P_r$

$= \{ (\lambda + \lambda^*)\hat{R} - 2\lambda\lambda^* + 2$  $\} P_r$ (17.46)

$R^3 P_r = (\lambda + \lambda^*)\hat{R}^2 - 4\lambda\lambda^* \hat{R} + 4(\lambda + \lambda^*)$  P_r (17.47)

We have used (17.11) invariance, (**??**), and the symmetry identity

 $= 0\,.$ (17.48)

Eliminating the extra invariant tensor in (17.47) by (17.46) we find that R satisfies a cubic equation symmetric under interchange $\lambda \leftrightarrow \lambda^*$

$$0 = (\mathbf{R} - (\lambda + \lambda^*))(\mathbf{R} - 2\lambda)(\mathbf{R} - 2\lambda^*)P_r\,, \qquad (17.49)$$

so the eigenvalues of $\mathbf{R}$ on the six subspaces of $\square \otimes \blacksquare$ are

$$\{\lambda_{\square}, \lambda_{\blacksquare}, \lambda_{\boxminus}, \lambda_5, \lambda_{\blacksquare}, \lambda_{\square\blacksquare}\} = \{1 - \lambda, 1/2, 1/2 - \lambda, 1/6, 2\lambda, 2\lambda^*\}.$$

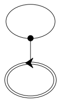
As in the preceding section, this leads to decomposition of the remainder subspace P_r into three subspaces:

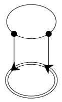
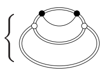
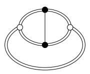
$$P_5 = -\frac{1}{(\lambda - \lambda^*)^2}(\mathbf{R} - 2\lambda)(\mathbf{R} - 2\lambda^*)P_r \qquad (17.50)$$

$$P_{\blacksquare} = \frac{1}{2(\lambda - \lambda^*)^2}(\mathbf{R} - (\lambda + \lambda^*))(\mathbf{R} - 2\lambda^*)P_r \qquad (17.51)$$

$$P_{\square\blacksquare} = \frac{1}{2(\lambda - \lambda^*)^2}(\mathbf{R} - (\lambda + \lambda^*))(\mathbf{R} - 2\lambda)P_r \qquad (17.52)$$

Dimension formulas of appendix A require that we evaluate

$\mathrm{tr}\,\mathbf{1} = N d_{\boxminus}\,, \qquad \mathrm{tr}\,R =$  $= 0$

$\mathrm{tr}\,R^2 =$  $= 2 \{$  $-$  $\} = 2(1 - \lambda) d_{\blacksquare}$ (17.53)

Substituting into (A.8) we obtain the dimensions of the three reps

$$d_5 = \frac{27(m-15)(2m-15)(m-8)(2m-9)(5m-24)(5m-36)}{m^2(m+3)(m+12)}$$

$$d_{\blacksquare} = \frac{5(m-5)(2m-15)(m-6)^2(m-8)(5m-36)}{m^3(m+3)(m+6)}(36 - m)$$

$$d_{\square\blacksquare} = \frac{5120(m-5)(2m-15)(m-6)^2(m-9)(2m-9)}{m^3(m+6)(m+12)}\,. \qquad (17.54)$$

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m	5	8	9	10	12	15	18	24	30	36
N	0	3	8	14	28	52	78	133	190	248
d_5	0	0	1	7	56	273	650	1,463	1,520	0
$d_{\blacksquare}$	0	-3	0	64	700	4,096	11,648	40,755	87,040	147,250
$d_{\square\blacksquare}$	0	0	27	189	1,701	10,829	34,749	152,152	392,445	779,247

Table 17.2 All solutions of Diophantine condition (17.54); a bogus $m = 30$ solution still survives this set of conditions. This solution will be eliminated by (**??**) which says that it does not exist for the F_4 subgroup of E_8.

We see that nothing significant is gained beyond the decomposition of $\mathrm{Sym}^3 A$ of the preceding section; we have recovered reps (**??**), (**??**). Representation $P_{\square\blacksquare}$ as from (17.52), (17.54) is new, but yields no new Diophantine condition.

If we consider reduction of $\square \times \square\square$ Kronecker product instead, the only difference is that (17.53) changes to $2(1 - \lambda^*)d_{\square\square}$, and we obtain 2 conjugate reps corresponding to $m/6 \leftrightarrow 6/m$ exchange;

$$d_6 = \qquad (17.55)$$

$$d_7 = \qquad (17.56)$$

17.4 DIOPHANTINE CONDITIONS

The Diophantine condition (17.14) and (17.19) are satisfied only for m = 8,9,10,12,18,20,24,30 and 36. $m = 30$ is a bogus solution, which does not survive further Diophantine condition (17.37).

The solutions of the above Diophantine conditions are listed in table 17.2. The formulas (17.54)-(17.54) yield, upon substitution of N, λ and λ^*, the correct Clebsch-Gordan series for all members of the E_8 family, table 17.2.

17.5 RECENT PROGRESS

The construction of the E_8 family described here was initiated [36, 37] in 1975 and an outline was published [41] in 1981. The derivation presented here, based on the assumption of no quartic primitive invariant (see fig. 16.1), was inspired by the work of S. Okubo [120].

17.5.1 Related literature

E. Angelopoulos is credited by M. El Houari [60] for obtaining in an unpublished paper (written around 1987) the Cartan classification using only methods of tensor calculus. Inspired by Angelopoulos and ref. [36], in his thesis M. El Houari applies a combination of tensorial and diagrammatic methods to the problem of classification of simple Lie algebras and superalgebras [60]. As *Algebras, Groups, and Geometries* journal does not practice proofreading (all references are of form

[?,?,?]), precise intellectual antecedents to this work are not easily traced. In a recent publication [7] E. Angelopoulos uses the spectrum of the Casimir operator acting on $A^{\otimes 2}$ to classify Lie algebras, and, *inter alia* also obtains the E_8 family of this chapter within a same class of Lie algebras.

17.5.2 Conjectures of Deligne

In a 1995 paper Deligne [49] attributed to Vogel [153] the observation that for the 5 exceptional groups the antisymmetric $\wedge^2 A$ and the symmetric $\mathrm{Sym}^2 A$ adjoint rep (tensor products $P_{\square} + P_{\substack{\square\\\square}}$ and $P_{\bullet} + P_{\square\square} + P_{\blacksquare}$ in (17.6), respectively) can be decomposed into irreducible reps in a uniform way, and that their dimensions and casimirs are rational functions of Vogel's parameter a, related to the parameter m of (17.13) by

$$a = \frac{1}{m-6}\,. \tag{17.57}$$

Here a is $a = \Phi(\alpha, \alpha)$, where α is the largest weight of the rep, and Φ the canonical bilinear form for the Lie algebra, in the notation of Bourbaki. Deligne conjectured that for A_1, A_2, G_2, F_4, E_6, E_7 and E_8, the dimensions of higher tensor reps $\otimes^k A$ could likewise be expressed as rational functions of parameter a.

The conjecture was checked on computer by Cohen and de Man [28] for dimensions and quadratic casimirs for all reps up to $\otimes^4 A$. They note that "miraculously for all these rational functions both numerator and denominator factor in $Q[a]$ as a product of linear factors". The miracle is perhaps explained by the method of decomposing symmetric subspaces outlined in this chapter.

Cohen and de Man have also observed that D_4 should be added to Deligne's list, in agreement with our definition of the E_8 family, consisting of A_1, A_2, G_2, D_4, F_4, E_6, E_7 and E_8. Their algebra goes way beyond the results given in this chapter, all which were obtained by paper and pencil birdtrack computations performed on trains while commuting between Gothenburg and Copenhagen. Cohen and de Man give formulas for in all 25 reps, 7 of which are also computed here.

In the context of this chapter $-a = \lambda^* = 1/6 - \lambda$ is the symmetric space eigenvalue of the invariant tensor $\mathbf{Q}$, in (17.13). The role of the tensor $\mathbf{Q}$ is to split the traceless symmetric subspace, and its overall scale is arbitrary. In this chapter scale was fixed by setting the adjoint rep quadratic casimir equal to unity, $C_A = 1$ in (17.4). Deligne [49] and Cohen and de Man [28] fix the scale of their λ, λ^* by setting $\lambda + \lambda^* = 1$, so their dimension formulas are stated in terms of a parameter related to the λ used here by $\lambda_{CdM} = 6\lambda$. Further "translation dictionary" relations: (17.37) is their A, (17.37) is their Y_3^*, (17.37) is their C^*.

They refer to the interchange of the roots $\lambda \leftrightarrow \lambda^*$ as "involution ".

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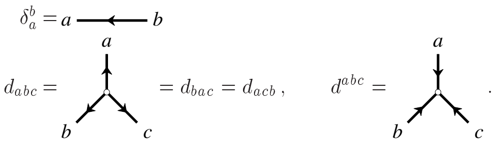
Chapter Eighteen

E_6 family of invariance groups

In this chapter, we determine all invariance groups whose primitive invariant tensors are δ^a_b and fully symmetric d_{abc}, d^{abc}. The reduction of $\otimes V^2$ space yields a rule for evaluation of the loop contraction of 4 d-invariants (18.9). The reduction of $V \otimes \overline{V}$ yields the first Diophantine condition (18.13) on the allowed dimensions of the defining rep. The reduction of $\otimes V^3$ tensors is straightforward, but the reduction of $A \otimes V$ space yields the second Diophantine condition (d_4 in table 18.4) and limits the defining rep dimension to $n \leq 27$. The solutions of the two Diophantine conditions form the E_6 *family* consisting of $E_6, A_5, A_2 + A_2$ and A_2. For the most interesting $E_6, n = 27$ case, the cubic casimir (18.45) vanishes. This property of E_6 enables us to evaluate loop contractions of 6 d-invariants (18.38), reduce $\otimes A^2$ tensors, table 18.5, and investigate relations among the higher order casimirs of E_6 in sect. 18.8. In sect. 18.7, we introduce a Young tableaux notation for any rep of E_6 and exemplify its use in construction of the Clebsch-Gordan series, table 18.6.

18.1 REDUCTION OF TWO-INDEX TENSORS

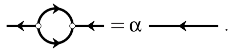
By assumption, the primitive invariants set that we shall study here is



(18.1)

Irreducibility of the defining n-dimensional rep implies

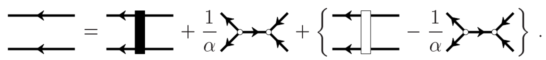
$$d_{abc}d^{bcd} = \alpha\delta^d_a$$



(18.2)

The value of α depends on the normalization convention. For example, Freudenthal [72] takes $\alpha = 5/2$. Konstein [103] and Kephart [92] take $\alpha = 10$. We find it convenient to set it to $\alpha = 1$.

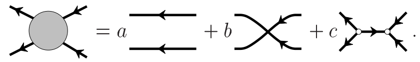
We can immediately write the Clebsch-Gordon series for the 2-index tensors. The symmetric subspace in (9.4) is reduced by the $d_{abc}d_{cde}$ invariant:



(18.3)

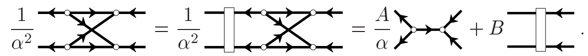
The rep dimensions and Dynkin indices are given in table 18.1.

By the primitiveness assumption, any $V^2 \otimes \overline{V}^2$ invariant is a linear combination of all tree invariants which can be constructed from the primitives:



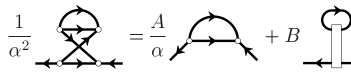
(18.4)

In particular,



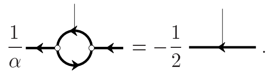
(18.5)

One relation on constants A, B follows from a contraction with δ^b_a:



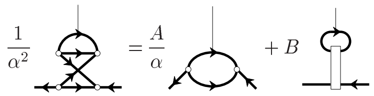
$$1 = A + B\frac{n+1}{2}\,.$$

The other relation follows from the invariance condition (6.55) on d_{abc}:



(18.6)

Contracting (18.5) with $(T_i)^b_a$, we obtain

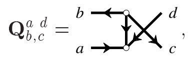


$$\frac{1}{4} = -\frac{A}{2} + \frac{B}{2}$$

$$A = -\frac{n-3}{2(n+3)}, \qquad B = \frac{3}{n+3}. \tag{18.7}$$

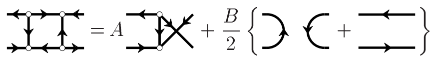
18.2 MIXED TWO-INDEX TENSORS

Let us apply the above result to the reduction of $V \otimes \overline{V}$ tensors. As always, they split into a singlet and a traceless part (9.45). However, now there exists an additional invariant matrix



(18.8)

which, according to (18.5) and (18.7) satisfies the characteristic equation



$$\mathbf{Q}^2 = -\frac{1}{2}\frac{n-3}{n+3}\mathbf{Q} + \frac{1}{2}\frac{3}{n+3}(\mathbf{T} + 1). \tag{18.9}$$

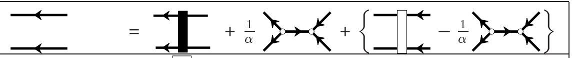
Projection operators							
Dynkin labels	$\square\times\square$	=	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	+	$\blacksquare$	+	$\square\square$
E_6	$(000010)\times(000010)$	=	(000100)	+	(100000)	+	(000020)
A_5	$(00010)\times(00010)$	=	(00101)	+	(01000)	+	(00020)
A_2	$(02)\times(02)$	=	(12)	+	(20)	+	(04)
Dimensions	n^2	=	$\frac{n(n-1)}{2}$	+	n	+	$\frac{n(n-1)}{2}$
E_6	27^2	=	351	+	27	+	351
A_5	15^2	=	105	+	15	+	105
A_2+A_2	9^2	=	36	+	9	+	36
A_2	6^2	=	15	+	6	+	15
Dynkin indices	$2n\ell$	=	$(n-2)\ell$	+	ℓ	+	$(n+1)\ell$
E_6	$2\cdot 27\cdot\frac{1}{4}$	=	$\frac{25}{4}$	+	$\frac{1}{4}$	+	7
A_5	$2\cdot 15\cdot\frac{1}{3}$	=	$\frac{13}{3}$	+	$\frac{1}{3}$	+	$\frac{16}{3}$
A_2+A_2	$2\cdot 9\cdot\frac{1}{2}$	=	$\frac{7}{2}$	+	$\frac{1}{2}$	+	5
A_2	$2\cdot 6\cdot\frac{5}{6}$	=	$\frac{10}{3}$	+	$\frac{5}{6}$	+	$\frac{35}{6}$

Table 18.1 E_6 family Clebsch-Gordon series for $\otimes V^2$. The defining rep Dynkin index ℓ is computed in (18.14).

On the traceless $V\otimes\overline{V}$ subspace, the characteristic equation for $\mathbf{Q}$ takes form

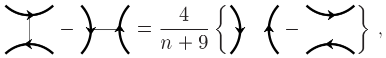
$$P_2\left(\mathbf{Q}+\frac{1}{2}\right)\left(\mathbf{Q}-\frac{3}{n+3}\right)=0\,, \tag{18.10}$$

where P_2 is the traceless projection operator (9.45). The associated projection operators (3.45) are

$$P_A=\frac{\mathbf{Q}-\frac{3}{n+3}}{-\frac{1}{2}-\frac{3}{n+3}}P_2,\qquad P_B=\frac{\mathbf{Q}+\frac{1}{2}}{\frac{3}{n+3}+\frac{1}{2}}P_2. \tag{18.11}$$

Their birdtracks form and their dimensions are given in table 18.2.

P_A, the projection operator associated with the eigenvalue $-\frac{1}{2}$, is the adjoint rep projection operator, as it satisfies the invariance condition (18.6). To compute the dimension of the adjoint rep, we use the relation



(18.12)

which follows trivially from the form of the projection operator P_A in table 18.2. The dimension is computed by taking trace (3.49),

$N=$  $=\dfrac{4n(n-1)}{n+9}\,.$ (18.13)

The 6-j coefficient, needed for the evaluation of the Dynkin index (7.26), can also

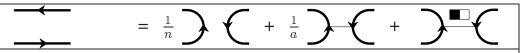
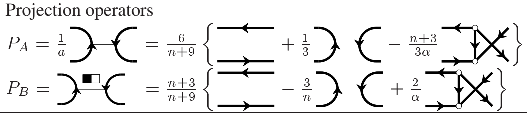
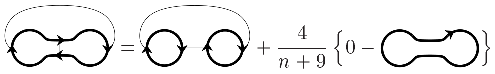
							
Dynkin labels	$\square\times\blacksquare$	=	$\bullet$	+	$\boxtimes$	+	$\blacksquare\square$
E_6	$(000010)\times(100000)$	=	(000000)	+	(000001)	+	(100010)
A_5	$(00010)\times(01000)$	=	(00000)	+	(10001)	+	(01010)
A_2	$(02)\times(20)$	=	(00)	+	(11)	+	(22)
Dimensions	n^2	=	1	+	$\frac{4n(n-1)}{n+9}$	+	$\frac{(n+3)^2(n-1)}{n+9}$
E_6	27^2	=	1	+	78	+	650
A_5	15^2	=	1	+	35	+	189
A_2+A_2	9^2	=	1	+	16	+	64
A_2	6^2	=	1	+	8	+	27
Dynkin indices	$2n\ell$	=	0	+	1	+	$\frac{2(n+3)^2}{n+9}\ell$
E_6	$2\cdot 27\cdot\frac{1}{4}$	=	0	+	1	+	$50\cdot\frac{1}{4}$
A_5	$2\cdot 15\cdot\frac{1}{3}$	=	0	+	1	+	$27\cdot\frac{1}{3}$
A_2+A_2	$2\cdot 9\cdot\frac{1}{2}$	=	0	+	1	+	$16\cdot\frac{1}{2}$
A_2	$2\cdot 6\cdot\frac{5}{6}$	=	0	+	1	+	$\frac{54}{6}$
Projection operators							

Table 18.2 E_6 family Clebsch-Gordon series for $V\otimes\overline{V}$. The defining rep Dynkin index ℓ is computed in (18.14).

be evaluated by substituting (18.12) into



$$=N\left(1-\frac{4}{n+9}\right)\,.$$

The Dynkin index for the E_6 family is

$$\ell=\frac{1}{6}\frac{n+9}{n-3}\,. \tag{18.14}$$

18.3 DIOPHANTINE CONDITIONS AND THE E_6 FAMILY

The expressions for the dimensions of various reps (see tables in this chapter) are ratios of polynomials in n, the dimension of the defining rep. As the dimension of a rep should be a non-negative integer, these relations are the Diophantine conditions on the allowed values of n. The dimension of the adjoint rep (18.13) is one such condition; the dimension of λ_4 from table 18.4 another. Furthermore, the positivity of the dimension λ_4 restricts the solutions to $n \le 27$. This leaves us with 6 solutions $n = 3, 6, 9, 15, 21, 27$. As we shall show in chapter 21, of these solutions only $n = 21$ is spurious - the remaining five solutions are realized as the E_6 row of the magic triangle, fig. 1.1.

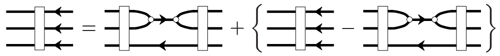
In the Cartan notation, the corresponding Lie algebras are A_2, $A_2 + A_2$, A_5 and E_6. We do not need to prove this, as for E_6 Springer has already proved the existence of a cubic invariant, satisfying the relations required by our construction, and for the remaining Lie algebras the cubic invariant is easily constructed, see sect. 18.9. We call these invariance groups the E_6 *family* and list the corresponding dimensions, Dynkin labels and Dynkin indices in the tables of this chapter.

18.4 THREE-INDEX TENSORS

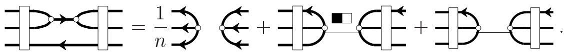
The $\otimes V^3$ tensor subspaces of $SU(n)$, listed in table 9.3, are decomposed by invariant matrices constructed from the cubic primitive d_{abc} in the following manner.

18.4.1 Fully symmetric $\otimes V^3$ tensors

We substitute expansion from table 18.1 into the symmetric projection operator

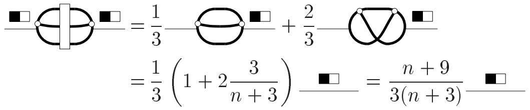


The $V \otimes \overline{V}$ subspace is decomposed by the expansion of table 18.2:



(18.15)

The last term vanishes by the invariance condition (6.55). To get the correct projector operator normalization for the second term, we compute

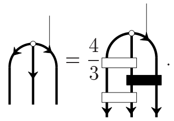


(18.16)

Here, the second term is given by the ■□-subspace eigenvalue (18.10) of the invariant matrix $\mathbf{Q}$ from (18.8). The resulting decomposition is given in table 18.3.

18.4.2 Mixed symmetry $\otimes V^3$ tensors

The invariant $d_{abe}(T_i)_c^e$ satisfies



(18.17)

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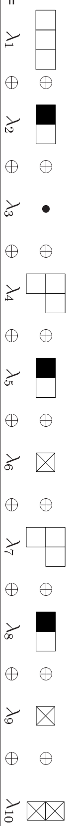
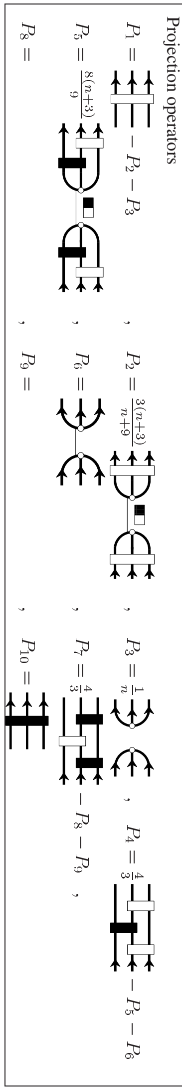


E_6 family:	$\otimes V^3 =$	λ_1	$\oplus\ \lambda_2$	$\oplus\ \lambda_3$	$\oplus\ \lambda_4$	$\oplus\ \lambda_5$	$\oplus\ \lambda_6$	$\oplus\ \lambda_7$	$\oplus\ \lambda_8$	$\oplus\ \lambda_9$	$\oplus\ \lambda_{10}$
Dynkin labels				0							
E_6	$(000010)^3 =$	(000030)	$+(100010)$	$+(000000)$	$+(000110)$	$+(100010)$	$+(000001)$	$+(000110)$	$+(100010)$	$+(000001)$	$+(001000)$
A_5	$(00010)^3 =$	(00030)	$+\ (01010)$	$+\ (00000)$	$+\ (00111)$	$+$	$+\ (10001)$	$+$			$+(00200)+(01002)$
A_2	$(02)^3 =$	(06)	$+\ (22)$	$+\ (00)$	$+\ (14)$	$+$	$+\ (11)$	$+$			$+\ (03)+(30)$
Dimensions	$n^3 =$	$\frac{n(n^2-1)(n+6)}{6(n+9)}$	$+\ \frac{(n+3)^2(n-1)}{n+9}$	$+\ 1$	$+\ \frac{(n^2-1)(n-3)}{3}$	$+$	$+\ \frac{4n(n-1)}{n+9}$	$+$			$+\frac{n(n-1)(n-2)}{6}$
E_6	$27^3 =$	3003	+ 650	+ 1	+ 5834	+	+ 78	+			+ 2925
A_5	$15^3 =$	490	+ 189	+ 1	+ 896	+	+ 35	+			+ 175 + 280
A_2+A_2	$9^3 =$	100	+ 64	+ 1	+ 160	+	+ 16	+			+ 84
A_2	$6^3 =$	28	+ 27	+ 1	+ 35	+	+ 8	+			+ $10+\overline{10}$
Dynkin indices				0							
E_6				0							
A_5				0							
A_2+A_2				0							
A_2				0							

Projection operators



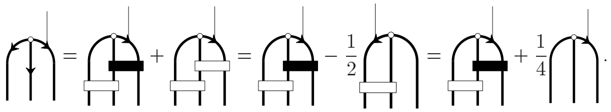
$P_1 = \ldots - P_2 - P_3$, $P_2 = \frac{3(n+3)}{n+9} \ldots$, $P_3 = \frac{1}{n} \ldots$, $P_4 = \frac{4}{3} \ldots - P_5 - P_6$

$P_5 = \frac{8(n+3)}{9} \ldots$, $P_6 = \ldots$, $P_7 = \frac{4}{3} \ldots - P_8 - P_9$,

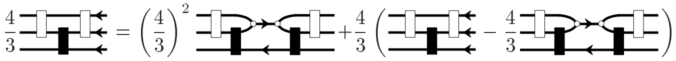
$P_8 =$, $P_9 =$, $P_{10} = \ldots$

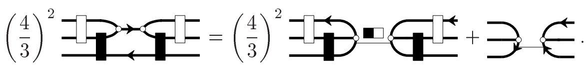
Table 18.3 E_6 family Clebsch-Gordan series for $\otimes V^3$. The dimensions and Dynkin indices of repeated reps are listed only once. ■□ is defined in table 18.2.

This follows from the invariance condition (6.55):



Hence, the adjoint subspace lies in the mixed symmetry subspace, projected by (9.10). Substituting expansions of tables 18.2 and 18.3, we obtain





(18.18)

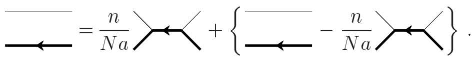
The corresponding decomposition is listed in table 18.3. The other mixed symmetry subspace from table 9.3 decomposes in the same way.

18.4.3 Fully antisymmetric $\otimes V^3$ tensors

All invariant matrices on $\otimes V^3 \to \otimes V^3$, constructed from d_{abc} primitives, are symmetric in at least a pair of indices. They vanish on the fully antisymmetric subspace, hence, the fully antisymmetric subspace in table 9.3 is irreducible for E_6.

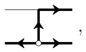
18.5 DEFINING $\otimes$ ADJOINT TENSORS

We turn next to the determination of the Clebsch-Gordan series for $V \otimes A$ reps. As always, this series contains the n-dimensional rep



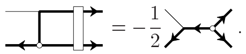
$1 = \quad P_1 \quad + \quad P_5$ (18.19)

Existence of the invariant tensor



(18.20)

implies that $V \otimes A$ also contains a projection onto the $\otimes V^2$ space. The symmetric rep in (18.3) does not contribute, as the d_{abc} invariance reduces (18.20) to a projection onto the V space:

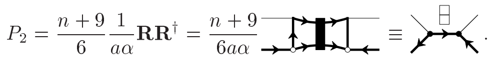


(18.21)

The antisymmetrized part of (18.20)

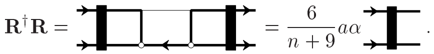
(18.22)

projects out the $\otimes V^2$ antisymmetric intermediate state, as in (18.3):



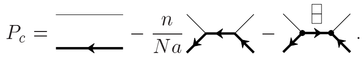
(18.23)

Here the normalization factor is evaluated by substituting the adjoint projection operator P_A (table 18.2) into



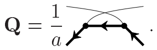
(18.24)

In this way, P_5 in (18.19) reduces to $P_5 = P_2 + P_c$,



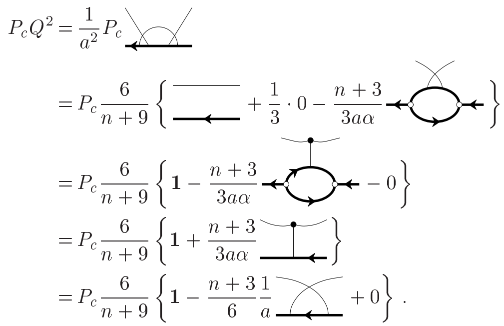
(18.25)

However, P_c subspace is also reducible, as there exists still another invariant matrix on $V \otimes A$ space:



(18.26)

We compute $\mathbf{Q}^2 P_c$ by substituting the adjoint projection operator and dropping the terms which belong to projections onto V and $\otimes V^2$ spaces:



(18.27)

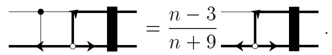
The resulting characteristic equation is surprisingly simple

$$P_c(\mathbf{Q}+1)\left(\mathbf{Q}-\frac{6}{n+9}\right)=0\,. \tag{18.28}$$

The associated projection operators and rep dimensions are listed in table 18.4. The rep λ_4 has dimension zero for $n = 27$, singling out the exceptional group $E_6(27)$.

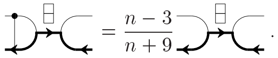
Vanishing dimension implies that the corresponding projection operator (4.20) vanishes identically. This could imply a relation between the contractions of primitives, such as the G_2 alternativity relation implied by the vanishing of (16.30). To investigate this possibility, we expand P_4 from table 18.4.

We start by using the invariance conditions and the adjoint projection operator P_A from table 18.2 to evaluate

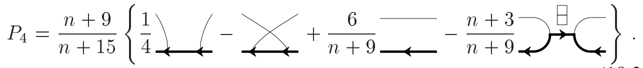


(18.29)

This yields

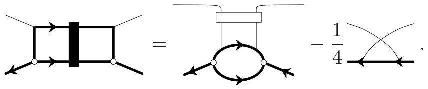


(18.30)



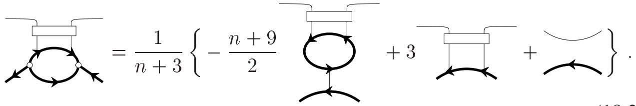
(18.31)

Next, motivated by the hindsight of the next section, we rewrite P_2 in terms of the cubic casimir (7.43). First we use invariance and Lie algebra (4.45) to derive relation



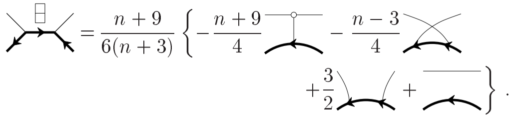
(18.32)

Next we use the adjoint projection operator (18.11) to replace the $d_{abc}d^{cde}$ pair in the first term



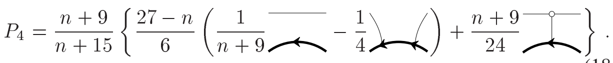
(18.33)

In terms of the cubic casimir (7.43), the P_2 projection operator is given by



(18.34)

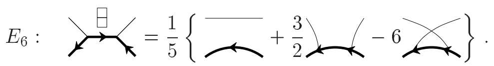
Substituting back into (18.31), we obtain



(18.35)

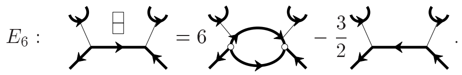
We shall show in the next section that the cubic casimir, in the last term, vanishes for $n = 27$. Hence, each term in this expansion vanishes separately for $n = 27$, and no new relation follows from the vanishing of P_4. Too bad.

However, the vanishing of the cubic casimir for $n = 27$ does lead to several important relations, special to the E_6 algebra. One of these is the reduction of the loop contraction of 6 d_{abc}'s. For E_6 (18.34) becomes



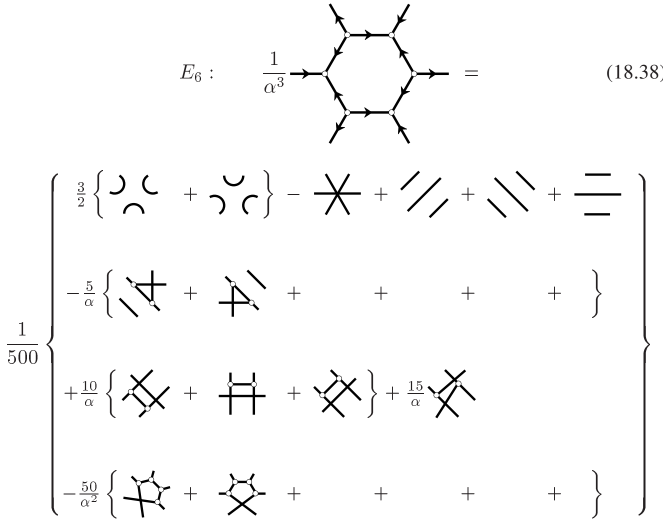
(18.36)

The left hand side of this equation is related to a loop of 6 d_{abc}'s (after substituting the adjoint projection operators):



(18.37)

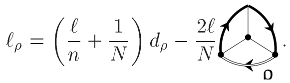
The right hand side of (18.36) contains no loop contractions. Substituting the adjoint operators in both sides of (18.36), we obtain a reduction formula for *loops of length 6*:



(18.38)

At the time of writing this report, we lack a proof that we can compute any scalar invariant built from d_{abc} contractions. However, the scalar invariants which we might be unable to compute are of very high order, bigger than anything listed in table 5.1, as their shortest loop must be of length eight or longer.

The Dynkin indices, in table 18.4, are computed using (7.28) with $\lambda =$ defining rep, $\mu =$ adjoint rep, $\rho = \lambda_3, \lambda_4$

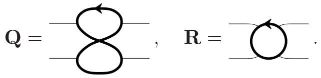


(18.39)

The value of the 6-j coefficient follows from (**??**), the eigenvalues of the exchange operator $\mathbf{Q}$.

18.6 TWO-INDEX ADJOINT TENSORS

$\otimes A^2$ has the usual starting decomposition (17.7). As in sect. 9.1, we study the index interchange and the index contractions invariants $\mathbf{Q}$ and $\mathbf{R}$:



(18.40)

	$A \otimes V$	=	λ_1	$\oplus$	λ_2	$\oplus$	λ_3	$\oplus$	λ_4
Dynkin labels		=		$\oplus$		$\oplus$			
E_6	(000001) × (000010)	=	(000010)	+	(010000)	+	(000011)		
A_5	(10001) × (00010)	=	(00010)	+	(10100)	+	(10011)	+	(00002)
A_2	(11) × (02)	=	(02)	+	(21)	+	(13)	+	(10)
Dimensions	nN	=	n	+	$\frac{n(n-1)}{2}$	+	$\frac{4n(n+1)(n-3)}{n+15}$	+	$\frac{n(n-1)(n-3)(27-n)}{2(n+9)(n+15)}$
E_6	$27 \cdot 78$	=	27	+	351	+	1728	+	0
A_5	$15 \cdot 35$	=	15	+	105	+	384	+	21
$A_2 + A_2$	$9 \cdot 16$	=	9	+	36	+	90	+	9
A_2	$6 \cdot 8$	=	6	+	15	+	24	+	3
Dynkin indices	$n + N\ell$	=	ℓ	+	$(n-2)\ell$	+	$\frac{5(n+1)(n+9)}{3(n+15)}$	+	$\frac{(n-5)(27-n)}{6(n+15)}$
E_6	$27 + \frac{78}{4}$	=	$\frac{1}{4}$	+	$\frac{25}{4}$	+	40	+	0
A_5	$15 + \frac{35}{3}$	=	$\frac{1}{3}$	+	$\frac{13}{3}$	+	$\frac{64}{3}$	+	$\frac{2}{3}$
$A_2 + A_2$	$9 + \frac{16}{2}$	=	$\frac{1}{2}$	+	$\frac{7}{2}$	+	$\frac{25}{2}$	+	$\frac{1}{2}$
A_2	$6 + \frac{8 \cdot 5}{6}$	=	$\frac{5}{6}$	+	$\frac{10}{3}$	+	$\frac{25}{3}$	+	$\frac{1}{6}$

Projection operators

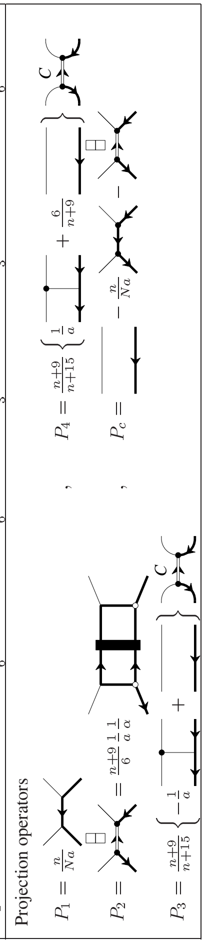
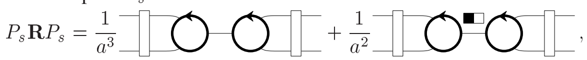


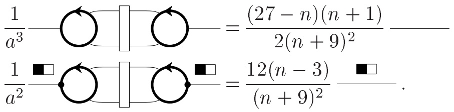
Table 18.4 E_6 family Clebsch-Gordan series for $A \otimes V$.

The decomposition induced by $\mathbf{R}$ follows from table 18.2; it decomposes the symmetric subspace P_s

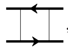


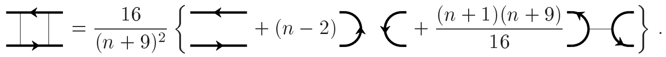
(18.41)

and, by (9.72), has no effect on the antisymmetric subspaces P_A, P_a. The corresponding projection operators are normalized by evaluating

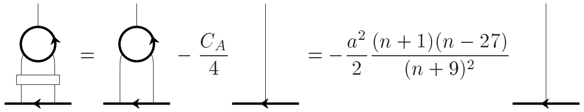


(18.42)

Such relations are evaluated by substituting the Clebsch-Gordan series of table 18.2 into , which yields

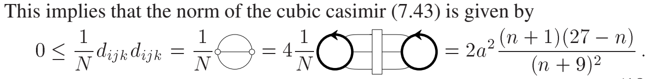


(18.42) then follows by substitution into



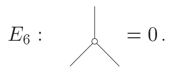
(18.43)

This implies that the norm of the cubic casimir (7.43) is given by



(18.44)

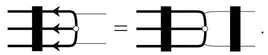
Positivity of the norm restricts $n \leq 27$. For E_6 $(n = 27)$, the cubic casimir vanishes identically



(18.45)

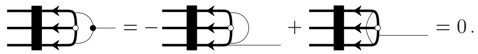
18.6.1 Reduction of antisymmetric three-index tensors

Consider the Clebsch-Gordan coefficient for projecting the antisymmetric subspace of $\otimes V^3$ onto $\otimes A^2$. By symmetry, it projects only onto the antisymmetric subspace of $\otimes A^2$:



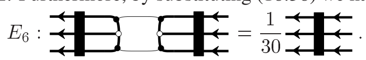
(18.46)

Furthermore, it does not contribute to the adjoint subspace:



(18.47)

That both terms vanish can easily be checked by substituting the adjoint projection operator, table 18.2. Furthermore, by substituting (18.38) we have (for $n = 27$)



(18.48)

This means that for E_6 reps ....and........are equivalent.

	$\otimes A^2 =$	λ_1	$\oplus$	λ_2	$\oplus$	λ_3	$\oplus$	λ_4	$\oplus$	λ_5	$\oplus$	λ_6	$\oplus$	λ_7
Dynkin labels		$\bullet$		+		■□	+	⊠⊠	+	⊠	+		⊠⊠	
E_6	$(000001)^2 =$	(000000)		+		(100010)	+	(000002)	+	(000001)	+		(00100)	
A_5	$(10001)^2 =$	(00000)	+	(10001)	+	(01010)	+	(20002)	+	(10001)	+	(01002)	+	(20010)
A_2	$(11)^2 =$	(00)	+	(11)		+		(22)	+	(11)	+	(03)	+	(30)
Dimensions	$N^2 =$	1	+	$N(1-\delta_{n,27})$	+	$\frac{(n+3)^2(n-1)}{n+9}$	+		+	N	+			
E_6	$78^2 =$	1	+	0	+	650	+	2430	+	78	+		2925	
A_5	$35^2 =$	1	+	35	+	189	+	405	+	35	+	280	+	280
$A_2 + A_2$	$16^2 =$	1	+	16	+		+		+	16	+	52	+	52
A_2	$8^2 =$	1	+	8	+	0	+	27	+	8	+	10	+	10
Dynkin indices														
E_6														
A_5														
$A_2 + A_2$														
A_2														

Projection operators for $E_6(n = 27)$:

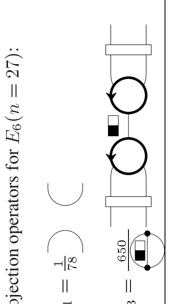
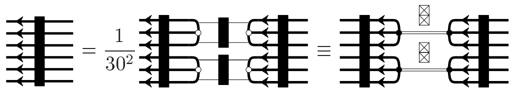


Table 18.5 E_6 family Clebsch-Gordan series of $\otimes A^2$.

18.7 DYNKIN LABELS AND YOUNG TABLEAUX FOR E_6

A rep of E_6 is characterized by 6 Dynkin labels $(a_1a_2a_3a_4a_5a_6)$. The corresponding Dynkin diagram is given in table 7.7. The relation of the Dynkin labels to the Young tableaux (see sect. 7.9) is less obvious than in the case of $SU(n)$, $SO(n)$ and $Sp(n)$ groups, because for E_6 they correspond to tensors made traceless also with respect to the cubic invariant d_{abc}.

The first three labels a_1, a_2, a_3 have the same significance as for the $SU(n)$ Young tableaux. a_1 counts the number of (not antisymmetrized) contravariant indices (columns of one box ■). a_2 counts the number of antisymmetrized contravariant index pairs (columns of 2 boxes ■■). a_3 is the number of antisymmetrized covariant index triples. That is all as expected, as the symmetric invariant d_{abc} cannot project anything from the antisymmetric subspaces. That is why the antisymmetric reps in table 18.1 and table 18.3 have the same dimension as for $SU(27)$. However, according to .... , an antisymmetric contravariant index triple is equivalent to an antisymmetric pair of adjoint indices. Hence, contrary to the $SU(n)$ intuition, this rep is *real*. We can use the Clebsch-Gordan coefficients from (18.48) to turn any set of $3p$ antisymmetrized contravariant indices into p adjoint antisymmetric index pairs. For example, for $p = 2$ we have



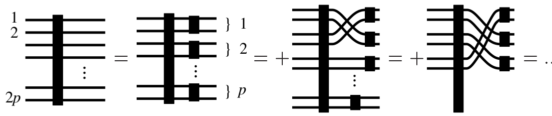
(18.49)

Hence, a column of more than 2 boxes is always reduced modulo 3 to a_3 antisymmetric adjoint pairs (in the above example $a_3 = p$), which we shall denote by columns of 2 crossed boxes ⊠⊠.

In the same fashion, the antisymmetric covariant index n-tuples contribute to a_3, the number of antisymmetric adjoint pairs ⊠⊠, a_4 antisymmetrized covariant index pairs ⊠⊠, and a_5 (not antisymmetrized) covariant indices □.

Finally, taking a trace of a covariant-contravariant index pair implies removing both a singlet *and* an adjoint rep. We shall denote the adjoint rep by ⊠. The number of (not antisymmetrized) adjoint indices is given by a_6. For example, an $SU(n)$ tensor $x^a_b \in V \otimes \overline{V}$ decomposes into 3 reps of table 18.2. The first one is the singlet (000000), which we denote by $\bullet$. The second one is the adjoint subspace $(0000001) =$ ⊠. The reminder is labelled by the number of covariant indices $a_1 = 1$, and contravariant indices $a_5 = 1$, yielding $(100010) =$ ■□ rep.

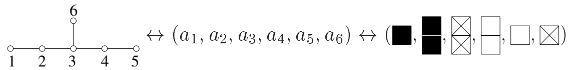
Any set of $2p$ antisymmetrized adjoint indices is equivalent to p *symmetrized* pairs by the identity



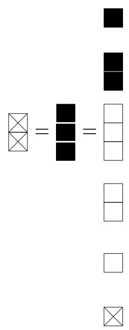
(18.50)

This reduces any column of 3 ⊠⊠⊠ or more antisymmetric indices. We conclude that any irreducible E_6 tensor can, therefore, be specified by 6 numbers $a_1, a_2, ...a_6$.

An E_6 tensor is made irreducible by projecting out all invariant subspaces. We do this by identifying all invariant tensors with right indices and symmetries and constructing the corresponding projection operators, as exemplified by table 18.1 through 18.5. If we are interested only in identifying the terms in a Clebsch-Gordan series, this can be quickly done by listing all possible non-vanishing invariant projections (many candidates vanish by symmetry or the invariance conditions) and checking whether their dimensions (from the Patera-Sankoff tables [128]) add up. Examples are given in table 18.6. Mnemonically, we can summarize the correspondence between the irreducible E_6 tensors and the Dynkin labels by



(18.51)



$a_1 =$ number of not antisymmetrized contravariant indices

$a_2 =$ number of antisymmetrized contravariant pairs

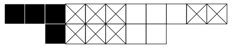
$a_3 =$ number of antisymmetrized adjoint index pairs

$a_4 =$ number of antisymmetrized covariant pairs

$a_5 =$ number of not antisymmetrized covariant indices

$a_6 =$ number of not antisymmetrized adjoint indices

For example, the Young tableau for the rep (2,1,3,2,1,2) can be drawn as



(18.52)

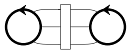
The difference in the number of the covariant and contravariant indices

$$a_1 + 2a_2 - 2a_4 - a_5 \pmod 3 \tag{18.53}$$

is called *triality*. Modulo 3 arises because of the conversion of antisymmetric triplets into the real antisymmetric adjoint pairs by (18.48). The triality is a useful check of correctness of a Clebsch-Gordan series, as all subspaces in the series must have

18.8 CASIMIRS FOR E_6

In table 7.1 we have listed the orders of independent casimirs for E_6 as 2, 5, 6, 8, 9, 12. Here we shall use our construction of $E_6(27)$ to partially prove this statement. By the hermiticity of T_i, the fully symmetric tensor d_{ijk} from (18.44) is real, and



$$= (d_{ijk})^2 \geq 0. \tag{18.54}$$

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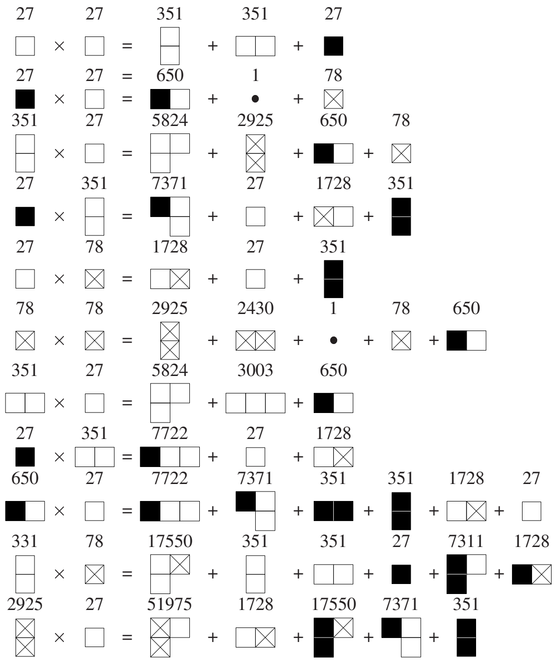
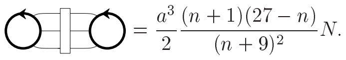


Table 18.6 Examples of the E_6 Clebsch-Gordan series in terms of the Young tableaux. Various terms in the expansion correspond to projections on various subspaces, indicated by the Clebsch-Gordan coefficients listed on the right. See table 18.1 through 18.5 for explicit projection operators.

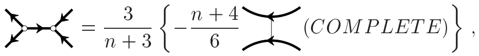
By (18.44), this equals



(18.55)

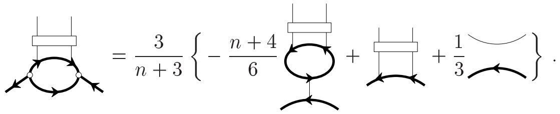
The cubic casimir d_{ijk} *vanishes identically* for E_6.

Next we prove that the *quartic casimir* for E_6 is reducible. From the expression for the adjoint rep projection operator we have



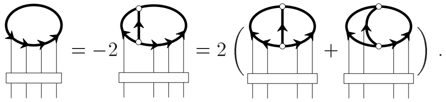
(18.56)

which yields



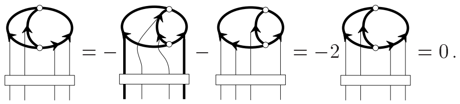
(18.57)

Now the quartic casimir. By the invariance (6.55)



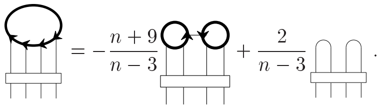
(18.58)

The second term vanishes by the same invariance condition



(18.59)

Substituting (18.33), we obtain



(18.60)

For E_6 the cubic casimir vanishes, and consequently the quartic casimir is reducible:

$$E_6 : \mathrm{tr}\, X^4 = \frac{1}{12}(\mathrm{tr}\, X^2)^2 \,. \qquad (18.61)$$

The *quintic* casimir $\mathrm{tr}\, X^5$ must be irreducible, as it cannot be expressed as a power of $\mathrm{tr}\, X^2$. (To check that it does not vanish identically, the reader should compute the analogue of (**??**)).

We leave it as an exercise to the reader to prove that $\mathrm{tr}\, X^6$ is irreducible.

To prove the reducibility of $\mathrm{tr}\, X^7$, we first streamline our notation by introducing the E_6 defining rep analogue of the determinant (6.47)

$$(A, B, C) \equiv \frac{1}{\alpha} d_{abc} d^{a'b'c'} A^{\prime a}_{a} B^{\prime b}_{b} C^{\prime c}_{c} \,, \qquad (18.62)$$

with A, B, C arbitrary $[n \times n]$ matrices. The invariance condition (6.55) for d_{abc} implies

$$(T_i A, B, C) + (A, T_i B, C) + (A, B, T_i C) = 0 \qquad (18.63)$$

$$(T_i A, A, A) = 0 \,. \qquad (18.64)$$

With the normalization condition (18.3), the septic casimir can be written as

$$\mathrm{tr}\, X^7 = (X^7, 1, 1) \,. \qquad (18.65)$$

We manipulate this expression by means of the invariance condition (6.55)

$$\begin{aligned}(X^7, 1, 1) &= -2(X^6, X, 1) = 2(X^5, X^2, 1) = 2(X^5, X, X) \\ &= \ldots = 7(X^3, X^3, X) + 6(X^3, X^2, X^2) \qquad (18.66)\end{aligned}$$

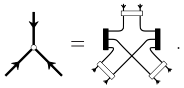
The second term vanishes by invariance (6.55). Substituting (**??**) into the first term, we obtain a formula that reduces the septic casimir:

$$(X^7, 1, 1) = -14(X^4, X^3, 1) = 14\{(X^5, X^2, 1) + (X^4, X^2, X^1)\} \,. \qquad (18.67)$$

18.9 SUBGROUPS OF E_6

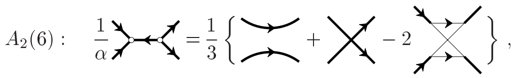
Why is $A_2(6)$ in the E_6 family?

The symmetric 2-index rep (9.2) of $SU(3)$ is 6-dimensional. The symmetric cubic invariant (18.2) can be constructed using a pair of Levi-Civita tensors

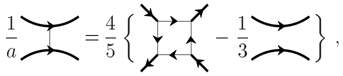


(18.68)

Contractions of several d_{abc}'s can be reduced using the projection operator properties (6.28) of Levi-Civita tensors, yielding expressions such as



(18.69)

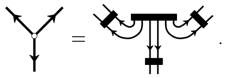


(18.70)

etc. The reader can check that, for example, the Springer relation (18.72) is satisfied.

Why is $A_5(15)$ in the E_6 family?

The antisymmetric 2-index rep (9.3) of $A_5 = SU(6)$ is 15-dimensional. The symmetric cubic invariant (18.2) is constructed using the Levi-Civita invariant (6.27) for $SU(6)$:



(18.71)

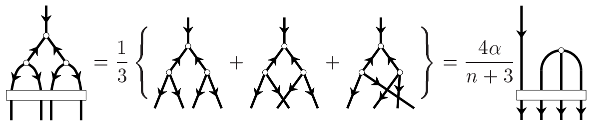
The reader is invited to check the correctness of the primitivity assumption (18.5). All other results of this chapter then follow.

Is $A_2 + A_2(9)$ in the E_6 family?

Exercise for the reader: unravel the $A_2 + A_2$ 9-dimensional rep, construct the d_{abc} invariant.

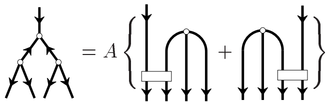
18.10 SPRINGER RELATION

Substituting P_A into the invariance condition (6.55) for d_{abc}, one obtains the *Springer relation [146, 147]*



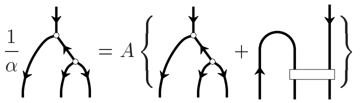
(18.72)

The Springer relation can be used to eliminate one of the 3 possible contractions of 3 d_{abc}'s. For the G_2 family it was possible to reduce *any* contraction of 3 f_{abc}'s by (16.15); however, a single chain of 3 d_{abc}'s *cannot* be reducible. If it were, symmetry would dictate a reduction relation of the form



(18.73)

Contracting with d_{abc} one finds that contractions of pairs of d_{abc}'s should also be reducible.



(18.74)

Contractions of this relation with d_{abc} and δ^a_b yields $n = 1$, *ie.* reduction relation (18.73) can be satisfied only by a trivial 1-dimensional defining rep.

18.10.1 Springer's construction of E_6

In the preceding sections we have given a self-contained derivation of the E_6 family, in a form unfamiliar to a handful of living experts. Here, we shall translate our results into more established notations, and identify those relations which have already been given by other authors.

Consider the exceptional simple Jordan algebra A of Hermitian $[3 \times 3]$ matrices x with octonian matrix elements (Freudenthal [71, 72]), and its dual $\overline{A}$ (complex conjugate of A). Following Springer [146, 147], define products

$$(\overline{x}, y) = \mathrm{tr}\,(\overline{x}y) \qquad (18.75)$$

$$x \times y = \overline{z} \qquad (18.76)$$

$$3 < x, y, z > = (x \times y, z)\,,$$

and assume that they satisfy

$$(x \times x) \times (x \times x) = < x, x, x > x\,. \qquad (18.77)$$

The nonassociative multiplication rule for elements x can be written in a basis $x = x_a \mathbf{e}_a$. Expanding $x, \overline{x}$ in (**??**), we chose a normalization

$$(\mathbf{e}_a, \mathbf{e}^b) = \delta^b_a\,, \qquad a, b = 1, 2, ..., 27 \qquad (18.78)$$

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and define

$$\mathbf{e}_a \times \mathbf{e}_b = d_{abc}\mathbf{e}^c. \qquad (18.79)$$

Substituting into (**??**), we obtain (**??**), with $\alpha = \frac{5}{2}$. Freudenthal and Springer prove that (**??**) is satisfied if d_{abc} is related to the usual Jordan product

$$\mathbf{e}_a \cdot \mathbf{e}_b = \hat{d}_{abc}\mathbf{e}_c, \qquad (18.80)$$

by

$$d_{abc} \equiv \hat{d}_{abc} - \frac{1}{2}[\delta_{ab}\mathrm{tr}\,(\mathbf{e}_c) + \delta_{ac}\mathrm{tr}\,(\mathbf{e}_b) + \delta_{bc}\mathrm{tr}\,(\mathbf{e}_a)] - \frac{1}{2}\mathrm{tr}\,(\mathbf{e}_a)\mathrm{tr}\,(\mathbf{e}_b)\mathrm{tr}\,(\mathbf{e}_c)]\,.$$

$E_6(27)$ is the group of isomorphisms which leave $(\overline{x}, y) = \delta^b_a x^a y_b$ and $< x, y, z > = d^{abc}x_a y_b z_c$ invariant. The derivation was constructed by Freudenthal (equation (1.21) in ref. [71]):

$$Dz \equiv < x, \overline{y} > z = 2\overline{y} \times (x \times z) - \frac{1}{2} < \overline{y}, z > x - \frac{1}{6} < x, \overline{y} > z\,.$$

Substituting (**??**), we obtain the projection operator (**??**):

$$(Dz)_d = -3x_a y^b P^{ac}_{bd} z_c. \qquad (18.81)$$

The object $< z, \overline{y} >$ considered by Freudenthal is in our notation and the above factor -3 is the normalization (**??**), Freudenthal's equation (1.26). The invariance of the x-product is given by Freudenthal as

$$< x, x \times x > = 0.$$

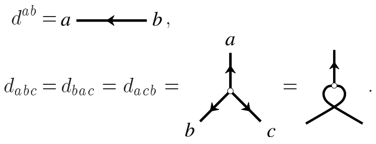
Substituting (**??**) we obtain (**??**) for d_{abc}.

Chapter Nineteen

F_4 family of invariance groups

In this chapter we classify and construct all invariance groups whose primitive invariant tensors are a symmetric bilinear d_{ab}, and a symmetric cubic d_{abc}, satisfying the relation (19.15). The results are summarized in table **??**.

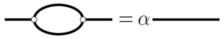
Take as primitives a symmetric quadratic invariant d_{ab} and a symmetric cubic invariant d_{abc}. As explained in chapter 12, we can use d_{ab} to lower all indices. In the birdtrack notation, we drop the open circles denoting d^{ab}, and we drop arrows on all lines:

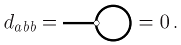


(19.1)

The defining n-dimensional rep is by assumption irreducible, so

$$d_{abc}d_{bcd} = \alpha\delta_{ad}$$

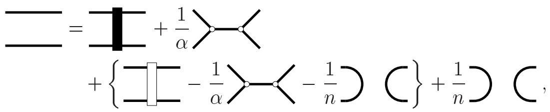

(19.2)


(19.3)

Were (19.3) nonvanishing, we could use  to project out a 1-dimensional subspace. The value of α depends on the normalization convention (Schafer [143] takes $\alpha = 7/3$).

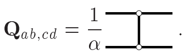
19.1 TWO-INDEX TENSORS

d_{abc} is a Clebsch-Gordan coefficient for $V \otimes V \to V$, so $V \otimes V$ space is decomposed into at least four subspaces:



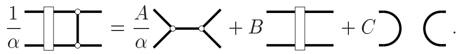
$$1 = A + P_2 + P_3 + P_1 . \qquad (19.4)$$

We turn next to the decompositions induced by the invariant matrix



(19.5)

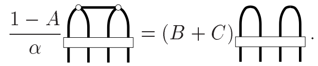
We shall assume that $\mathbf{Q}$ does not decompose the symmetric subspace, *ie.* that its symmetrized projection can be expressed as


(19.6)

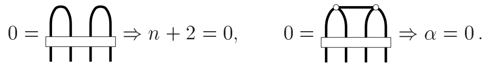
Together with the list of primitives (19.1), this assumption *defines* the F_4 family. This corresponds to the assumption (16.3) in the construction of G_2. We have not been able to construct the F_4 family without this assumption.

Invariance groups with primitives d_{ab}, d_{abc} which do not satisfy (19.6) also exist. The most familiar example is the adjoint rep of $SU(n)$, $n \geq 4$, where d_{abc} is the Gell-Mann symmetric tensor (9.79).

Symmetrizing (19.6) in all legs, we obtain

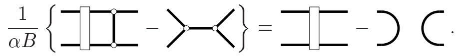

(19.7)

Neither of the tensors can vanish, as contractions with δ's would lead to

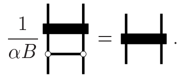


(19.8)

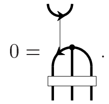
If the coefficients were to vanish, $1 - A = B + C = 0$, we would have


(19.9)

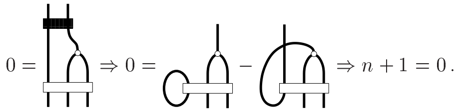
Antisymmetrizing the top two legs, we find that in this case also the antisymmetric part of the invariant matrix $\mathbf{Q}$ (19.5) is reducible:


(19.10)

This would imply that the adjoint rep of $SO(n)$ would also be the adjoint rep for the invariance group of d_{abc}. However, the invariance condition

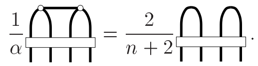

(19.11)

cannot be satisfied for any positive dimension n :

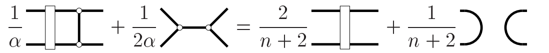


(19.12)

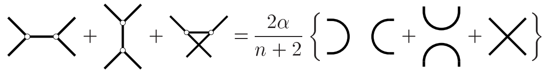
Hence, the coefficients in (19.7) are non-vanishing and are fixed by tracing with δ_{ab}:


(19.13)

Expanding the symmetrization operator, we can write this relation as


(19.14)

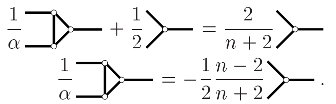
(this fixes $A = -1/2$, $B = 2/(n+2)$, $C = 1/(n+2)$ in (19.6)), or as



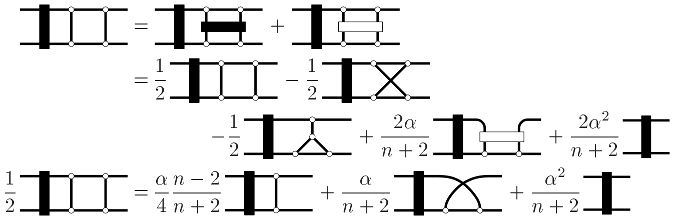
$$d_{abe}d_{ecd} + d_{ade}d_{ebc} + d_{ace}d_{ebd} = \frac{2\alpha}{n+2}(\delta_{ab}\delta_{cd} + \delta_{ad}\delta_{bc} + \delta_{ac}\delta_{bd})\,. \quad (19.15)$$

In sect. 19.4, we shall show that this relation can be interpreted as the characteristic equation for $[3\times 3]$ octonian matrices. This is the *defining relation* for the F_4 family.

The eigenvalue of the invariant matrix $\mathbf{Q}$ on the n-dimensional subspace can now be computed from (19.14)

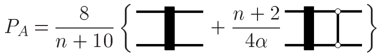

(19.16)

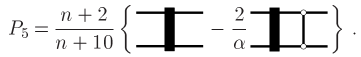
Let us now turn to the action of the invariant matrix $\mathbf{Q}$ on the antisymmetric subspace in (19.4). We evaluate $\mathbf{Q}^2$ with the help of the characteristic equation (19.14):



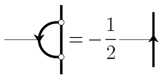
$$0 = A\left(\mathbf{Q}^2 - \frac{1}{2}\frac{n-6}{n+2}\mathbf{Q} - \frac{2}{n+2}\right)\,. \quad (19.17)$$

The roots are $\lambda_A = -1/2$, $\lambda_5 = 4/(n+2)$, and the associated projectors are


(19.18)


(19.19)

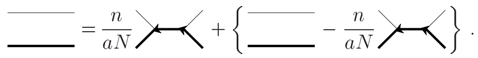
The dimensions and Dynkin indices are listed in P_A is the projector for the adjoint rep, as it satisfies the invariance condition (19.11):



$$P_A\mathbf{Q} = -\frac{1}{2}P_A\,. \quad (19.20)$$

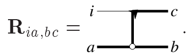
19.2 DEFINING $\otimes$ ADJOINT TENSORS

$V \otimes A$ space always contains the defining rep:

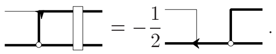


$$1 = P_6 + P_7\,. \quad (19.21)$$

We can use d_{abc} and $(T_i)_{ab}$ to project a $V \otimes V$ subspace from $V \otimes A$:

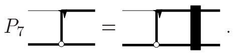

(19.22)

By the invariance condition (19.11), $\mathbf{R}$ acting on the symmetrized $V \otimes V$ subspace projects it on V

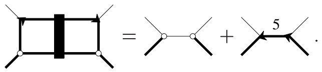

(19.23)

Hence, $\mathbf{R}$ maps the P_7 subspace only onto the antisymmetrized $V \otimes V$:

$$P_7\mathbf{R} = \mathbf{R}A$$

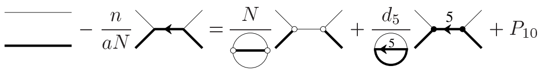

(19.24)

The $V \otimes V$ space was decomposed in the preceding section. Using (19.18) amd (19.19), we have

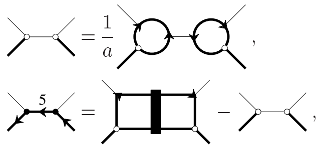

(19.25)

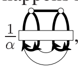
The P_7 space can now be decomposed as

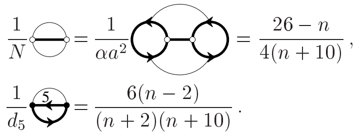
$$P_7 = P_8 + P_9 + P_{11}$$


(19.26)

Here,


(19.27)

and the normalization factors are the usual normalizations for 3-vertices. An interesting thing happens in evaluating the normalization for the subspace: substituting (19.18) into , we obtain

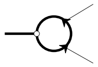

(19.28)

The normalization factor is a sum of squares of real numbers:

$$= \frac{1}{\alpha a^2}\sum_{i,j,a}\left[(T_i)_{bc}d_{acd}(T_j)_{db}\right]^2 \geq 0 . \qquad (19.29)$$

Hence, either $n = 26$ or $n < 26$ the corresponding Clebsch-Gordan coefficients are identically zero:

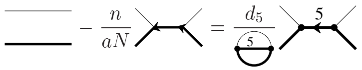
$$n = 26 :$$



$$= 0 , \qquad (19.30)$$

and P_7 subspace in (19.26) does not contain the adjoint rep, *ie.* (19.26) is replaced by

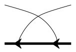
$$n = 26 :$$



$$+ P_{10} . \qquad (19.31)$$

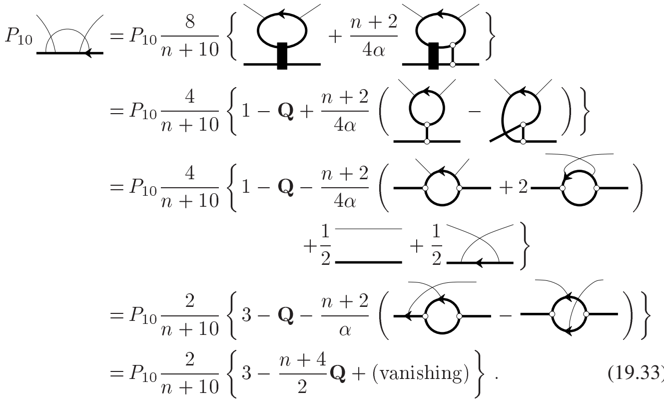
Another invariant matrix on $V \otimes A$ space can be formed from two $(T_i)_{ab}$ generators:

$$\mathbf{Q} =$$



(19.32)

We compute $P_{10}\mathbf{Q}^2$ by substituting the adjoint projection operator by (19.18), using the characteristic equation (19.14) and the invariance condition, and dropping the contributions to the subspaces already removed from P_{10}:



(19.33)

Hence, $\mathbf{Q}^2$ satisfies a characteristic equation

$$0 = P_{10}\left(\mathbf{Q}^2 + \frac{n+4}{n+10}\mathbf{Q} - \frac{6}{n+10}\right) , \qquad (19.34)$$

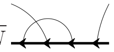
with roots $\alpha_{11} = -1, \alpha_{12} = 6/(n+10)$, and the corresponding projection operators

$$P_{11} = P_{10}\frac{n+10}{n+16}\left(\frac{6}{n+10} - \mathbf{Q}\right) ,$$

$$P_{12} = P_{10}\frac{n+10}{n+16}(1+\mathbf{Q}) . \qquad (19.35)$$

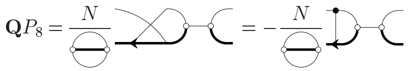
To use these expressions, we also need to evaluate the eigenvalues of the invariant matrix $\mathbf{Q}$ on subspaces P_6, P_8 and P_9 :

$$\mathbf{Q}P_6 = \frac{n}{aN}$$



$$= \left(\frac{N}{n} - \frac{C_A}{2}\right)P_6 = \frac{1}{2}P_6 . \qquad (19.36)$$

(It is somewhat surprising that this eigenvalue does not depend on the dimension n.)



$$= -\frac{N}{2n}P_8 = -\frac{3(n-2)}{2(n+10)}P_8$$

$$\mathbf{Q}P_q = -\frac{n-8}{n+10}P_q \qquad (19.37)$$

These relations are valid for any n.

Now we can evaluate the dimensions of subspaces P_{11}, P_{12}. We obtain for $n < 26$

$$d_{11} = \mathrm{tr}\, P_{11} = \frac{n(n-2)(n-5)(14-n)}{2(n+10)(n+16)}$$

$$d_{12} = \mathrm{tr}\, P_{12} = \frac{3n(n+1)(n-5)}{n+16} \qquad (19.38)$$

A new miracle has occurred: only $n = 26$ and $n \leq 14$ are allowed. However, $d_{12} < 0$ for $n < 5$ does not exclude the $n = 2$ solution, as in that case the adjoint rep is identically zero, and $V \otimes A$ decomposition is meaningless.

For $n = 26$, P_{10} is defined by (19.31), (the adjoint rep does not contribute), and the dimensions are given by

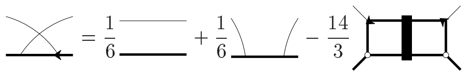
$$n = 26 : \qquad d_{11} = 0, \qquad d_{12} = 1053 . \qquad (19.39)$$

If a dimension is zero, the corresponding projector operator vanishes identically, and we have a relation between invariants

$$0 = P_{11} = P_{10}(1/6 - \mathbf{Q}) = (1 - P_6 - P_9)(1/6 - \mathbf{Q}) . \qquad (19.40)$$

ubstituting the eigenvalues of $\mathbf{Q}$, we obtain a special F_4 relation

$$n = 26 :$$

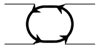


(19.41)

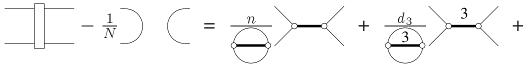
Hence, for $n = 26$ (F_4 Lie algebra) the two invariants, R in (19.24) and Q in (19.32), are not independent.

19.3 TWO-INDEX ADJOINT TENSORS

$\otimes A^2$ always decomposes into at least four reps (??). We consider first the $V \otimes V$ intermediate states

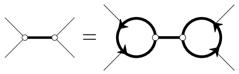
$$\mathbf{R} =$$  . (19.42)

The symmetric $V \otimes V$ intermediate states resolve the symmetric $A \otimes A$ space into:

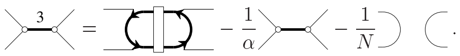
 $+ \; P_{15}$,

$$= \quad P_{13} \quad + \quad P_{14} \quad + P_{15} \; . \tag{19.43}$$

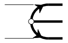
Here, the first projector is defined by (19.27)

 . (19.44)

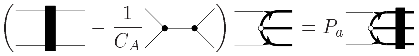
By (19.30) it vanishes for $n = 26$. The P_{14} is defined by

 . (19.45)

We consider next the $\otimes V^3$ intermediate states induced by the invariant

$$\mathbf{Q} =$$  . (19.46)

It is easily checked that due to the invariance condition (19.11), the only interesting mapping induced by $\mathbf{Q}$ is the antisymmetric $\otimes A^2 A \rightarrow$ antisymmetric $\otimes V^3$

$$P_a \mathbf{Q} =$$  (19.47)

19.4 JORDAN ALGEBRA AND $F_4(26)$

Consider the exceptional simple Jordan algebra of traceless Hermitian [3×3] matrices x with octonion matrix elements (Freudenthal [72], Schafer [143]). The nonassociative multiplication rule for elements x can be written, in a basis $x = x_a \mathbf{e}_a$, as

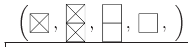
$$\mathbf{e}_a \mathbf{e}_b = \mathbf{e}_b \mathbf{e}_a = \frac{\delta_{ab}}{3} \mathbf{I} + \mathbf{d_{abc}} \mathbf{e_c}$$
$$a, b, c \in \{1, 2, \ldots, 26\} \,, \tag{19.48}$$

where $\mathrm{tr}\,(\mathbf{e}_a) = 0$ and $\mathbf{I}$ is the [3×3] unit matrix. Traceless [3×3] matrices satisfy a characteristic equation

$$x^3 - \frac{1}{2} \mathrm{tr}\,(x^2) x - \frac{1}{3} \mathrm{tr}\,(x^3) \mathbf{I} = \mathbf{0} \,. \tag{19.49}$$

Substituting we obtain with normalization $\alpha = \frac{7}{3}$. Substituting into the Jordan identity (Schafer [143])

$$(xy)x^2 = x(yx^2) \,, \tag{19.50}$$



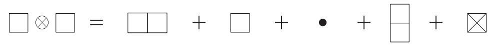
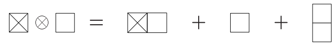
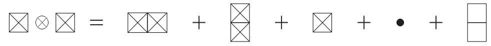
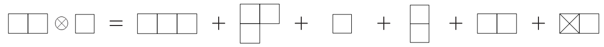
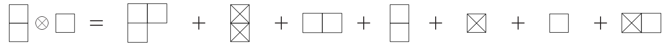
	$26^2 = 324 + 26 + 1 + 273 + 52$
	$52 \cdot 26 = 1053 + 26 + 273$
	$2704 = 52^2 = 1053 + 1274 + 52 + 1 + 324$
	$8424 = 324 \cdot 26 = 2652 + 4096 + 26 + 273 + 324 + 1053$
	$7098 = 273 \cdot 26 = 4096 + 1274 + 324 + 273 + 52 + 26 + 1053$

Table 19.1 TABLE FROM END OF F4FAMILY CHAPTER. NEEDS EDITING!

we obtain. It is interesting to note that the Jordan identity (which defines Jordan algebra in the way Jacobi identity defines Lie algebra) is a trivial consequence of (?!). $F_4(28)$ is the group of isomorphisms which leave forms $\mathrm{tr}\,(xy) = \delta_{ab} x_a x_b$ and $\mathrm{tr}\,(xyz) = d_{abc} x_a y_b z_c$ invariant. The derivation is given by Tits (see equation (28) in [151])

$$Dz = (xz)y - x(zy) \,. \qquad \text{Tits, equation (28)} \tag{19.51}$$

Substituting (19.48), we obtain the $n = 26$ case of the adjoint rep projection operator (19.18)

$$(Dz)_d = -3 x_a y_b \left(\frac{\delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd}}{9} + \frac{d_{bce} d_{ead} - d_{ace} d_{ebd}}{3} \right) z_c \,. \tag{19.52}$$

Chapter Twenty

E_7 family and its negative dimensional cousins

Parisi and Sourlas [127] have suggested that a Grassmann vector space of dimension n can be interpreted as an ordinary vector space of dimension $-n$. As we have seen in chapter 13, semi-simple Lie groups abound with examples in which $an \to -n$ substitution can be interpreted in this way. An early example were Penrose's binors [132], reps of $SU(2) = Sp(2)$ constructed as $SO(-2)$, and discussed here in chapter **??**. This is a special case of a general relation between $SO(n)$ and $Sp(-n)$ established in chapter 13; if symmetrizations and antisymmetrizations are interchanged, reps of $SO(n)$ become $Sp(-n)$ reps. Here we illustrate such relations by working out in detail an example motivated by Cremmer and Julia's discovery of a global E_7 symmetry in supergravity [32].

We shall extend the invariant length and volume which characterize the Lorentz group to a quadratic and a quartic supersymmetric invariant. The symmetry group of the Grassmann sector will turn out to be one of $SO(2)$, $SU(2)$, $SU(2) \times SU(2) \times SU(2)$, $Sp(6)$, $SU(6)$, $SO(12)$ or E_7, which also happens to be the list of possible global symmetries of extended supergravities.

We shall extend the Minkowski space into Grassmann dimensions by requiring that the invariants of $SO(4)$ (or $SO(3,1)$ - compactness plays no role in this analysis) become supersymmetric invariants. As shown in chapter 10, $SO(4)$ is the invariance group of the Kronecker delta $g_{\mu\nu}$ and the Levi-Civita tensor $\varepsilon_{\mu\nu\sigma\rho}$, hence, we are looking for the invariance group of the supersymmetric invariants

$$\begin{aligned} (x,y) &= g_{\mu\nu} x^\mu y^\nu \,, \\ (x,y,z,w) &= e_{\mu\nu\sigma\rho} x^\mu y^\nu z^\sigma w^\rho \,, \end{aligned} \tag{20.1}$$

where $\mu, \nu, \ldots = 4, 3, 2, 1, -1, -2, \ldots, -n$. Our secret motive for thinking of the Grassmann dimensions as $-n$ is that we think of the dimension as a trace, $n = \delta^\mu_\mu$, and in a Grassmann (or fermionic) world each trace carries a minus sign. For the quadratic invariant $g_{\mu\nu}$ alone the invariance group is the orthosymplectic $OSp(4, n)$. This group [89] is orthogonal in the bosonic dimensions and symplectic in the Grassmann dimensions, because if $g_{\mu\nu}$ is symmetric in the $\nu, \mu > 0$ indices, it must be antisymmetric in the $\nu, \mu < 0$ indices. In this way the supersymmetry ties in with the $SO(n) \sim Sp(-n)$ equivalence developed in chapter 13.

Following this line of reasoning, we assume that if the quartic invariant tensor $e_{\mu\nu\sigma\rho}$ is antisymmetric in ordinary dimensions, it is symmetric in the Grassmann dimensions. Our task is then to determine all groups which admit an antisymmetric quadratic invariant, together with a symmetric quartic invariant.

The resulting classification can be summarized by:

symmetric $d_{\mu\nu}$+ antisymmetric $f_{\mu\nu\sigma\rho}$:

$$(A_1 + A_1)(4), G_2(7), B_3(8), D_5(10)\,,$$

antisymmetric $f_{\mu\nu}$+ symmetric $d_{\mu\nu\sigma\rho}$:

$$SO(2), A_1(4), (A_1 + A_1 + A_1)(8), C_3(14), A_5(20), D_6(32), E_7(56)\,,$$

where the numbers in ( ) are the rep dimensions.

The second case generates a row of the Freudenthal magic triangle (fig. 1.1). In this chapter, we shall be using the matrix notation of Okubo [117], rather than the birdtrack notation used elsewhere in this text. From the supergravity point of view, it is important to note that the Grassmann space relatives of our $SO(4)$ world include E_7, $SO(12)$ and $SU(6)$ in the same reps as those discovered by Cremmer and Julia. Furthermore, it appears that *all* seven possible groups can be realized as global symmetries of the seven extended supergravities, if one vector multiplet is added to $N = 1, 2, 3$, and 4 extended supergravities.

Originally, the $n \to -n$ relations and the magic triangle arose as byproducts of an investigation of group-theoretic structure of gauge theories undertaken in ref. [36]. At the time they appeared to be mere mathematical curiosities, but since then their possible connection with Grassmann dimensions and supergravities has made them more intriguing.

In sect. 20.1 to sect. 20.3, we determine the groups which allow a symmetric quadratic invariant together with an antisymmetric quartic invariant. The end result of the analysis is two non-trivial Diophantine conditions together with the explicit projection operators for irreducible reps. In sect. 20.4, the analysis is repeated for an antisymmetric quadratic invariant together with a symmetric quartic invariant. We find the same Diophantine conditions, with dimension n replaced by $-n$, and the same projection operators, with symmetrizations and antisymmetrizations interchanged.

20.1 THE ANTISYMMETRIC QUARTIC INVARIANT

Add to the $SO(n)$ set of V^4 invariant tensors - identity $\mathbf{1}$ and flip σ from (6.2), the index contraction T - a fully antisymmetric invariant

$$f_{\mu\nu\rho\delta} = -f_{\nu\mu\rho\delta} = -f_{\mu\rho\nu\delta} = -f_{\mu\nu\delta\rho}\,. \tag{20.2}$$

The simplest $\left[n^2 \times n^2\right]$ matrix constructed from the new invariant is

$$\mathbf{E}_{\nu\rho}^{\mu\delta} = \delta^{\mu\varepsilon}\delta^{\delta\sigma} f_{\varepsilon\sigma\nu\rho}\,. \tag{20.3}$$

The $SO(n)$ multiplication rules $\sigma^2 = 1$, $\sigma\mathbf{T} = \mathbf{T}$, $\mathbf{T}^2 = n\mathbf{T}$ are now extended by

$$\mathbf{TE} = 0, \qquad \sigma\mathbf{E} = -\mathbf{E}\,. \tag{20.4}$$

The $\mathbf{E}$ invariant does not decompose the symmetric subspaces (10.11), (10.10):

$$P_1\mathbf{E} = 0 \qquad P_2\mathbf{E} = \frac{1}{2}(1+\sigma)\mathbf{E} = 0\,. \tag{20.5}$$

The $\mathbf{E}$ invariant can, however, decompose the V_3 subspace (10.12). As we wish to introduce one invariant at a time, we demand that no further independent $\left[n^2 \times n^2\right]$ invariant matrices can be constructed from $\mathbf{E}$. In particular, $\mathbf{E}^2$ is not independent:

$$\left(\mathbf{E}^2 + b\mathbf{E} + c\mathbf{1}\right) P_3 = 0\,. \tag{20.6}$$

This condition, incidentally, also insures that the $[n\times n]$ matrix $(\mathbf{E}^2)^{ij}_{kl}$ is proportional to unity:

$$(\mathbf{E}^2)^{ij}_{kl} = -c\frac{d_3}{n}\delta^j_k\,, \tag{20.7}$$

where $d_3 = n(n-1)/2$ is the dimension of the $SO(n)$ adjoint rep. Were this not true, distinct eigenvalues of $\mathbf{E}^2$ matrix would decompose the defining n-dimensional rep, contradicting our assumption that the defining rep is irreducible.

If the coefficients in (20.6) can be fixed, V_4 is split into the new adjoint rep subspace V_6 and the remainder V_7, by means of projection operators (3.48):

$$\begin{aligned} \text{adjoint:} \qquad & P_6 = \frac{\mathbf{E} - \alpha_7\mathbf{1}}{\alpha_6 - \alpha_7}P_3\,, \\ \text{antisymmetric:} \qquad & P_7 = \frac{\mathbf{E} - \alpha_6\mathbf{1}}{\alpha_7 - \alpha_6}P_3\,, \end{aligned} \tag{20.8}$$

where $\alpha_6 + \alpha_7 = -b$, $\alpha_6\alpha_7 = c$ are the roots of quadratic equation (20.6). The coefficient c is fixed by the scale of $\mathbf{E}$:

$$\operatorname{tr} E^2 + c d_3 = 0\,. \tag{20.9}$$

To fix the remaining coefficient b, introduce an index flip on the $\left[n^2 \times n^2\right]$ matrices:

$$\mathbf{F}(A)^{\mu\delta}_{\nu\rho} = A^{\delta\mu}_{\nu\rho}, \quad \mathbf{F}^2 = \mathbf{1}\,. \tag{20.10}$$

Combined with the invariant tensors listed above, the additional F multiplication rules are

$$\mathbf{F}(\mathbf{1}) = \mathbf{T}, \quad \mathbf{F}(\sigma) = \sigma, \quad \mathbf{F}(\mathbf{E}) = -\mathbf{E}\,. \tag{20.11}$$

It follows that

$$P_3\mathbf{F}(P_3) = \frac{1}{2}(\mathbf{1}-\sigma)\frac{1}{2}(\mathbf{T}-\sigma) = \frac{1}{2}P_3\,. \tag{20.12}$$

The characteristic equation (20.6) maps under $\mathbf{F}$ and P_3 projection into

$$P_3\left(\mathbf{F}(\mathbf{E}^2) - b\mathbf{E} + \frac{1}{2}c\mathbf{1}\right) = 0\,. \tag{20.13}$$

In particular, in the adjoint rep subspace V_6, using $P_6\mathbf{E} = \alpha_6 P_6$

$$P_6\left(\mathbf{F}(\mathbf{E}^2) + \alpha_6^2 - \frac{3}{2}\frac{\operatorname{tr}\mathbf{E}^2}{d_3}\right) = 0\,. \tag{20.14}$$

To compute $P_6\mathbf{F}(\mathbf{E}^2)$, one contracts the invariance condition (4.35) for $\mathbf{E}$ with another $\mathbf{E}$ matrix and uses the antisymmetry of $\mathbf{E}$ as well as (20.7). You might wonder, how we figured out such things? These calculations are a breeze in the birdtrack notation; but as people with more algebraic mindset find birdtracks repugnant, in

this chapter, for once, we hide our tracks behind conventional algebraic notation. The result is

$$P_6\mathbf{F}(\mathbf{E}^2) = \frac{1}{3}\frac{\operatorname{tr}\mathbf{E}^2}{n}P_6\,. \tag{20.15}$$

Now α_6, α_7 and the associated projection operators P_6, P_7 follow from (20.14) and (20.9):

$$\alpha_6 = \sqrt{\frac{\operatorname{tr}\mathbf{E}^2}{d_3}\frac{10-n}{6}}, \qquad \alpha_7 = -\sqrt{\frac{\operatorname{tr}\mathbf{E}^2}{d_3}\frac{6}{10-n}}\,, \tag{20.16}$$

$$\begin{aligned} \text{adjoint:}\ \ & P_6 = \sqrt{\frac{6(10-n)}{(16-n)^2}\frac{d_3}{\operatorname{tr}\mathbf{E}^2}}\,\mathbf{E} + \frac{6}{16-n}P_3\,, \\ \text{antisym:}\ \ & P_7 = -\sqrt{\frac{6(10-n)}{(16-n)^2}\frac{d_3}{\operatorname{tr}\mathbf{E}^2}}\,\mathbf{E} + \frac{10-n}{16-n}P_3\,, \end{aligned} \tag{20.17}$$

with the dimensions

$$d_6 = \operatorname{tr} P_6 = \frac{3n(n-1)}{16-n}\,, \qquad d_7 = \operatorname{tr} P_7 = \frac{n(n-1)(10-n)}{2(16-n)}\,. \tag{20.18}$$

This completes the decomposition $\otimes V^2 = V_1 \oplus V_5 \oplus V_6 \oplus V_7$. The new subspaces V_6, V_7 have integer dimension only for $n = 4, 6, 7, 8, 10$. However, the reduction of $\otimes V^3$ undertaken in the next section will eliminate the $n = 6$ possibility.

20.2 FURTHER DIOPHANTINE CONDITIONS

The reduction of the $\otimes V^2$ space, induced by the invariants δ^{ij}, and f_{ijkl}, has led to a very restrictive Diophantine condition (20.18). We shall now show that further Diophantine conditions follow from the reduction of higher product spaces $\otimes V^q$. As an example, we turn to the reduction of (adjoint) $\otimes$ (defining)=$V_6 \otimes V \subset \otimes V^3$. The tensor $x^\mu_{\nu\rho}$ is an element of the tensor space $V_6 \otimes V$ if

$$(P_6)^{\mu\nu'}_{\nu\mu'}x^{\mu'}_{\nu'\rho} = x^\mu_{\nu\rho}\,, \tag{20.19}$$

The simplest invariant matrices one can write down are

$$\begin{aligned} \text{identity:}\ \ & \mathbf{1}^{\mu}_{\nu\rho,}{}^{\alpha\gamma}_{\beta} = (P_6)^{\mu\alpha}_{\nu\beta}\delta^\gamma_\rho\,, \\ \text{defining rep:}\ \ & \mathbf{R}^{\mu}_{\nu\rho,}{}^{\alpha\gamma}_{\beta} = (P_6)^{\mu\sigma}_{\nu\rho}\delta^{\sigma'}_{\sigma}(P_6)^{\gamma\alpha}_{\sigma'\beta} \end{aligned} \tag{20.20}$$

The factor $\delta^{\sigma'}_{\sigma}$ in $\mathbf{R}$ is written out explicitly to indicate that $\mathbf{R}$ is a mapping $V_6 \otimes V \to V \to V_6 \otimes V$. The characteristic equation

$$\mathbf{R}^2 = \frac{d_6}{n}\mathbf{R} \tag{20.21}$$

yields projection operators

$$P_8 = \frac{n}{d_6}\mathbf{R}, \qquad P_9 = 1 - \frac{n}{d_6}\mathbf{R}\,. \tag{20.22}$$

Hence, $V_6 \otimes V = V_8 \oplus V_9$ with dimensions

$$d_8 = (P_8)_{\nu\rho,\mu}^{\mu}{}^{\nu\rho} = n\,, \qquad d_9 = \mathrm{tr}\,P_9 = n(d_6 - 1)\,. \tag{20.23}$$

The next invariant matrix we construct is an index permutation of $\mathbf{R}$:

$$\mathbf{Q}_{\nu\rho,\beta}^{\mu}{}^{\alpha\gamma} = (P_6)_{\nu\sigma}^{\mu\gamma}(P_6)_{\rho\beta}^{\sigma\alpha}\,. \tag{20.24}$$

In order to find the associated projector operators one has to compute

$$(\mathbf{Q}^2)_{\nu\rho,\beta}^{\mu}{}^{\alpha\gamma} = (P_6)_{\nu\sigma}^{\mu\rho'}(P_6)_{\rho\sigma'}^{\sigma\gamma}(P_6)_{\rho'\beta}^{\sigma'\alpha}\,.$$

This is achieved by substituting $(P_6)_{\rho\sigma'}^{\sigma\gamma}$ from (20.17) and using the invariance condition (4.35). The result is

$$\mathbf{Q}^2 = \frac{1}{2(\alpha_6 - \alpha_7)}\left\{(\alpha_6 + \alpha_7)Q - \alpha_6 P_8 - \alpha_7 \mathbf{1}\right\}\,. \tag{20.25}$$

The n-dimensional space V_9 is reducible by the roots

$$\alpha_{10} = \frac{\alpha_7}{\alpha_6 - \alpha_7} \qquad \alpha_{11} = \frac{1}{2} \tag{20.26}$$

of the characteristic equation:

$$P_9\left(\mathbf{Q}^2 - \frac{1}{2}\frac{\alpha_6 + \alpha_7}{\alpha_6 - \alpha_7}\mathbf{Q} + \frac{1}{2}\frac{\alpha_7}{\alpha_6 - \alpha_7}\mathbf{1}\right) = 0\,. \tag{20.27}$$

Substituting (20.16), we obtain the associated projection operators

$$P_{10} = \frac{2(16-n)}{28-n}\left(-\mathbf{Q} + \frac{1}{2}\mathbf{1}\right)P_9\,,$$
$$P_{11} = \frac{2(16-n)}{28-n}\left(\mathbf{Q} + \frac{6}{16-n}\mathbf{1}\right)P_9\,. \tag{20.28}$$

This completes the decomposition $V \otimes V_6 = V_8 \oplus V_{10} \oplus V_{11}$. To compute the dimensions of V_{10}, V_{11} subspaces, we need $\mathrm{tr}\,P_9\mathbf{Q}$:

$$\mathrm{tr}\,P_9\mathbf{Q} = -\frac{2n(2+n)}{16-n}\,. \tag{20.29}$$

Finally, we obtain

$$d_{10} = \mathrm{tr}\,P_{10} = \frac{3n(n+2)(n-4)}{28-n}\,,$$
$$d_{11} = \mathrm{tr}\,P_{11} = \frac{32n(n-1)(n+2)}{(16-n)(28-n)}\,. \tag{20.30}$$

The important aspect of these relations is that the denominators, and hence, the Diophantine conditions, are different from those in (20.18). It is easy to check that of the solutions to (20.18) $d = 4, 7, 8, 10$ are also solutions of the present Diophantine conditions. All solutions are summarized in table 20.1.

20.3 LIE ALGEBRA IDENTIFICATION

As we have shown, symmetric $\delta_{\mu\nu}$ together with antisymmetric $f_{\mu\nu\sigma\rho}$ invariants cannot be realized in dimensions other than $d = 4, 7, 8, 10$. But can they be realized at all? To verify that, one can turn to the tables of Lie algebras of ref. [128] and identify these four solutions.

GroupTheory December 8, 2002

Representation	Dimension	$A_1 + A_1$	G_2	B_3	D_5
V=defining	n	4	7	8	10
V_6=adjoint	$\frac{3n(n-1)}{16-n}$	3	14	21	45
V_7=antisym.	$\frac{n(n-1)(10-n)}{2(16-n)}$	3	7	7	0
V_5=symmetric	$\frac{(n+2)(n-1)}{2}$	9	27	35	54
V_{10}	$\frac{3n(n+2)(n-4)}{28-n}$	0	27	48	120
V_{11}	$\frac{32n(n-1)(n+2)}{(16-n)(28-n)}$	8	64	112	320

Table 20.1 Representation dimensions for the $SO(4)$ family of invariance groups.

20.3.1 $SO(4)$ or $A_1 + A_1$ algebra

The first solution, $d = 4$, is not a surprise; it was $SO(4)$, Minkowski or euclidean version, that motivated the whole project. The quartic invariant is the Levi-Civita tensor $\varepsilon_{\mu\nu\rho\sigma}$. Even so, the projectors constructed are interesting. Taking

$$E_{\nu\rho}^{\mu\delta} = g^{\mu\varepsilon}g^{\delta\rho}\varepsilon_{\varepsilon\sigma\nu\gamma}\,, \tag{20.31}$$

one can immediately calculate (20.6):

$$\mathbf{E}^2 = 4P_3\,. \tag{20.32}$$

The projectors (20.17) become

$$P_6 = \frac{1}{2}P_3 + \frac{1}{4}\mathbf{E}, \quad P_7 = \frac{1}{2}P_3 - \frac{1}{4}\mathbf{E}\,, \tag{20.33}$$

and the dimensions are $d_6 = d_7 = 3$. Also both P_6 and P_7 satisfy the invariance condition, the adjoint rep splits into two invariant subspaces. In this way, one shows that the Lie algebra of $SO(4)$ is the semi-simple $SU(2) + SU(2) = A_1 + A_1$. Furthermore, the projection operators are precisely the $\eta, \overline{\eta}$ symbols used by 't Hooft [87] to map self-dual and self-antidual $SO(4)$ antisymmetric tensors onto $SU(2)$ gauge group:

$$(P_6)_{\nu\rho}^{\mu\delta} = \frac{1}{4}\left(\delta_\rho^\mu\delta_\nu^\delta - g^{\mu\delta}g_{\nu\rho} + \varepsilon^{\mu\delta}{}_{\nu\rho}\right) = -\frac{1}{4}\eta_{a\nu}^{\mu}\eta_{a\rho}^{\delta}\,,$$
$$(P_7)_{\nu\rho}^{\mu\delta} = \frac{1}{4}\left(\delta_\rho^\mu\delta_\nu^\delta - g^{\mu\delta}g_{\nu\rho} - \varepsilon^{\mu\delta}{}_{\nu\rho}\right) = -\frac{1}{4}\overline{\eta}_{a\nu}^{\mu}\overline{\eta}_{a\rho}^{\delta}\,. \tag{20.34}$$

The only difference is that instead of using an index pair ${}_\nu^\mu$, t' Hooft indexes the adjoint spaces by $a = 1, 2, 3$. All identities, listed in the appendix of ref. [87], now follow from the relations of sect. 20.1.

Representation	Dynkin index	$A_1 + A_1$	G_2	B_3	D_5
V=defining	$\frac{16-n}{4(n+2)}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{8}$
V_6=adjoint	1	1	1	1	1
V_7=antisym.	$\frac{(10-n)(n-4)}{4(n+2)}$	0	$\frac{1}{4}$	$\frac{1}{5}$	0
V_5=symmetric	$\frac{1}{4}(16-n)$	3	$\frac{9}{4}$	2	$\frac{3}{2}$
V_{10}	$\frac{7(16-n)(n-4)}{4(28-n)}$	–	$\frac{9}{4}$	$\frac{14}{5}$	$\frac{7}{2}$
V_{11}	$\frac{8(2n+7)}{(28-n)}$	5	8	$\frac{46}{5}$	12

Table 20.2 Dynkin indices for the $SO(4)$ family of invariance groups.

20.3.2 Defining rep of G_2

The 7-dimensional rep of G_2 is a subgroup of $SO(7)$, so it has invariants δ_{ij} and $\varepsilon_{\mu\nu\delta\sigma\rho\alpha\beta}$. In addition, it has an antisymmetric cubic invariant [20, 36] $f_{\mu\nu\rho}$, the invariant that we interpret in (**??**) as the multiplication table for octonions. The quartic invariant we have inadvertently rediscovered is

$$f_{\mu\nu\rho\sigma} = \varepsilon_{\mu\nu\rho\sigma\alpha\beta\gamma} f^{\alpha\beta\gamma} \,. \tag{20.35}$$

Furthermore, for G_2 we have the identity (16.15) by which any chain of contractions of more than two $f_{\alpha\beta\gamma}$ can be reduced. Projection operators of sect. 20.1 and sect. 20.2 yield the G_2 Clebsch-Gordan series (16.12)

$$7 \otimes 7 = 1 \oplus 27 \oplus 14 \oplus 7 \,,$$
$$7 \otimes 14 = 7 \oplus 27 \oplus 64 \,.$$

20.3.3 $SO(7)$ 8-dimensional rep

We have not attempted to identify the quartic invariant in this case. However, all the rep dimensions (table 20.1), as well as their Dynkin indices (table 20.2), match B_3 reps listed in tables of Patera and Sankoff [128].

20.3.4 $SO(10)$ 10-dimensional rep

This is a trivial solution; $P_6 = P_3$ and $P_1 = 0$, so that there is no decomposition. The quartic invariant is

$$f_{\mu\nu\sigma\rho} = \varepsilon_{\mu\nu\sigma\rho\alpha\beta\gamma\delta\omega\xi} C_{\alpha\beta,\gamma\delta,\omega\xi} \equiv 0 \,, \tag{20.36}$$

where $C_{\alpha\beta,\gamma\delta,\omega\xi}$ are the $SO(7)$ Lie algebra structure constants.

This completes our discussion of the "bosonic" symmetric $g_{\mu\nu}$, antisymmetric $e_{\alpha\beta\gamma\delta}$ invariant tensors. We turn next to the "ferminic" case: antisymmetric $g_{\mu\nu}$, symmetric $e_{\alpha\beta\gamma\delta}$.

20.4 SYMMETRIC QUARTIC INVARIANT

We have established in chapter 12 that the invariance group of antisymmetric quadratic invariant $f_{\mu\nu}$ is $Sp(n), n$ even. We now add to the set of $Sp(n) \otimes V^4$ invariants (**??**) a symmetric 4-index tensor

$$d_{\mu\nu\rho\delta} = d_{\nu\mu\rho\delta} = d_{\mu\rho\nu\delta} = d_{\mu\nu\delta\rho} \,. \tag{20.37}$$

Again, most of the algebra is the same as in sect. 20.1. Equations (20.3) to (20.9) are the same. We redefine the index permutation (20.10) as

$$\mathbf{F}(A)^{\mu\delta}_{\nu\rho} = -A^{\delta\mu}_{\nu\rho} \,, \qquad \mathbf{F}^2 = 1 \,. \tag{20.38}$$

Continuing as in sect. **??**, we have

$$\mathbf{F}(1) = -\mathbf{T} \,, \qquad \mathbf{F}(\sigma) = \sigma \,, \qquad \mathbf{F}(\mathbf{E}) = -\mathbf{E} \,. \tag{20.39}$$

(20.13), (20.14) still apply, but the present redefinition of $\mathbf{F}$ flips sign in (20.15)

$$P_6(\mathbf{E}^2) = -\frac{1}{3}\frac{\mathrm{tr}\,\mathbf{E}^2}{n} P_6 \,. \tag{20.40}$$

This amounts to replacing $n \to -n$ in all remaining expressions

$$\text{adjoint}: \; P_6 = \sqrt{\frac{6(10+n)d_3}{(16+n)^2 \mathrm{tr}\,\mathbf{E}^2}}\mathbf{E} + \frac{6}{16+n}P_3 \,,$$

$$\text{symmetric}: \; P_7 = -\sqrt{\frac{6(10+n)d_3}{(16+n)^2 \mathrm{tr}\,\mathbf{E}^2}}\mathbf{E} + \frac{10+n}{16+n}P_3 \,, \tag{20.41}$$

$$d_6 = \frac{3n(n+1)}{16+n} = 3n - 45 + \frac{360}{8+\frac{1}{2}n} \,, d_7 = d_4 - d_6 \,. \tag{20.42}$$

There are 17 solutions to this Diophantine condition, but only 10 will survive the next one.

20.4.1 Further Diophantine conditions

Rewriting sect. 20.2 for an antisymmetric $f_{\mu\nu}$, symmetric $d_{\mu\nu\sigma\rho}$ is absolutely trivial, as these tensors never make an explicit appearance. The only subtlety is that for the reductions of Kronecker products of odd numbers of defining reps (in this case $\otimes V^3$), additional overall factors of -1 appear. For example, it is clear that the dimension of the defining subspace d_8 in (20.23) does not become negative; $n \to -n$ substitution propagates only through the expressions for α_6, α_7 and d_6. The dimension formulas (20.30) become

$$d_{10} = \frac{3n(n-2)(n+4)}{n+28} \,,$$
$$d_{11} = \frac{32n(n-2)(n+1)}{(n+16)(n+28)} \,. \tag{20.43}$$

Out of the 17 solutions to (20.42), 10 also satisfy this Diophantine condition; $d = 2, 4, 8, 14, 20, 32, 44, 56, 164, 224$. $d = 44, 164$ and 224 can be eliminated [37] by considering reductions along the columns of the Freudenthal magic square and proving that the resulting subgroups cannot be realized; consequently the groups that contain them cannot be realized either. Only the 7 solutions listed in table 20.3 have antisymmetric $f_{\mu\nu}$ and symmetric $d_{\mu\nu\rho\delta}$ invariants in the defining rep.

20.4.2 Lie algebra identification

It turns out that one does not have to work very hard to identify the series of solutions of the preceding section. $SO(2)$ is trivial, and there is extensive literature on the remaining solutions. Mathematicians study them because they form the third row of the (extended) Freudenthal magic square [72], and physicists study them because $E_7(56) \to SU(3)_c \times SU(6)$ once was one of the favored unified models [78]. The rep dimensions and the Dynkin indices listed in tables **??** and **??**gree with the above literature, as well as with the Lie algebra tables [128]. Here, we shall explain only, why E_7 is one of the solutions.

The construction of E_7, closest to the spirit of our endeavor, has been carried out by Brown [15, 163]. He considers a n-dimensional complex vector space V with properties

(i) V possesses a non-degenerate skew-symmetric bilinear form $\{x, y\} = f_{\mu\nu} x^\mu y^\nu$.

(ii) V possesses a symmetric four-linear form $q(x, y, z, w) = d_{\mu\nu\sigma\rho} x^\mu y^\nu z^\sigma w^\rho$.

(iii) If the ternary product $\mathbf{T}(x, y, z)$ is defined on V by $\{\mathbf{T}(x, y, z), w\} = q(x, y, z, w)$, then $3\{\mathbf{T}(x, x, y), \mathbf{T}(y, y, y)\} = \{x, y\} q(x, y, y, y)$.

The third property is nothing but the invariance condition (4.35) for $d_{\mu\nu\rho\delta}$ as can be verified by substituting P_6 from (20.41). Hence, our quadratic, quartic invariants fulfill all three properties assumed by Brown. He then proceeds to prove that the 56-dimensional rep of E_7 has the above properties and saves us from that labor.

20.5 THE EXTENDED SUPERGRAVITIES AND THE MAGIC TRIANGLE

The purpose of all this algebra has been to show that the extension of Minkowski space into a superspace can be a non-trivial enterprise. We now have an exhaustive classification, but are there any realizations of it? Surprisingly enough, every single entry in our classification appears to be realized as a global symmetry of an extended supergravity.

Cremmer and Julia [32] have discovered that in $N = 8$ (or $N = 7$) supergravity's 28 vectors, together with their 28 duals, form a $\underline{56}$ multiplet of a global E_7 symmetry. This is a global symmetry analogous to $SO(2)$ duality rotations of the doublet $(F_{\mu\nu}, F{*}_{\mu\nu})$ in $j^\mu = 0$ sourceless electrodynamics. The appearance of E_7 was quite unexpected; it was the first time an exceptional Lie group emerged as a physical symmetry, without having been inserted into a model by hand. While the

Representation	$SO(2)$	A_1	$A_1+A_1+A_1$	C_3	A_5	D_6		E_7		
V=defining	2	4	8	14	20	32	44	56	164	224
V_6=adjoint	1	3	9	21	35	66	99	133	451	630
V_7=symmetric	2	7	27	84	175	462	891	1463	13079	24570
V_5=antisym.	0	5	27	90	189	495	945	1539	13365	24975
V_{10}	0	6	48	216	540	1728	3696	6480	69741	134976
$V-11$	0	2	16	64	$70+70$	352	616	912	4059	5920
Dynkin indices:										
V=defining		$\frac{5}{2}$	1	$\frac{5}{8}$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{5}{14}$	$\frac{1}{3}$	$\frac{5}{18}$	$\frac{10}{37}$
V_6=adjoint		1	1	1	1	1	1	1	1	1
V_7=symmetric		14	9	9	10	$\frac{63}{5}$	$\frac{108}{7}$	$\frac{55}{3}$	$\frac{406}{9}$	$\frac{2233}{37}$
V_5=antisym.		5	6	$\frac{15}{2}$	9	12	15	18	45	60
V_{10}		$\frac{35}{4}$	14	$\frac{45}{2}$	$\frac{63}{2}$	$\frac{252}{5}$	70	90	$\frac{2205}{8}$	380
$V-11$		$\frac{1}{4}$	2	4	$\frac{11}{4}+\frac{11}{4}$	$\frac{38}{5}$	9	10	$\frac{107}{8}$	14

Table 20.3 Representation dimensions and Dynkin indices for the E_7 family of invariance groups.

Representation	$SO(2)$	A_1	$A_1+A_1+A_1$	C_3	A_5	D_6		E_7		
V=defining		$\frac{5}{2}$	1	$\frac{5}{8}$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{5}{14}$	$\frac{1}{3}$	$\frac{5}{18}$	$\frac{10}{37}$
V_6=adjoint		1	1	1	1	1	1	1	1	1
V_7=symmetric		14	9	9	10	$\frac{63}{5}$	$\frac{108}{7}$	$\frac{55}{3}$	$\frac{406}{9}$	$\frac{2233}{37}$
V_5=antisym.		5	6	$\frac{15}{2}$	9	12	15	18	45	60
V_{10}		$\frac{35}{4}$	14	$\frac{45}{2}$	$\frac{63}{2}$	$\frac{252}{5}$	70	90	$\frac{2205}{8}$	380
$V-11$		$\frac{1}{4}$	2	4	$\frac{11}{4}+\frac{11}{4}$	$\frac{38}{5}$	9	10	$\frac{107}{8}$	14

Table 20.4 Dynkin indices for the E_7 family of invariance groups.

classification we have obtained here does not explain why this happens, it suggests that there is a deep connection between the extended supergravities and the exceptional Lie algebras. To establish this connection, observe that Cremmer and Julia's $N = 7, 6, 5$ global symmetry groups $E_7, SO(12), SU(6)$ are included in the present classification. Furthermore, vectors plus their duals form multiplets of dimension 56, 32, 20, so they belong to the defining reps in our classification. For $N \leq 4$ extended supergravities, the numbers of vectors do not match the dimensions of the defining reps. However, Poul Howe has pointed out that if one adds one vector multiplet, the numbers match up, and $N = 1, 2, \ldots, 7$ extended supergravities exhaust the present classification. These observations are summarized in table 5 of ref. [41].

The present classification is a row of the magic triangle, fig. 1.1. This is an extension of Freudenthal's magic square, an octonionic construction of exceptional Lie algebras. The remaining rows are obtained [37] by applying the methods of this monograph to various kinds of quadratic and cubic invariants, while the columns are subgroup chains. In this context, the Diophantine condition (20.42) is one of a family of Diophantine conditions discussed in chapter **??**. They all follow from formulas for the dimension of the adjoint rep of form

$$N = \frac{1}{3}(k-6)(l-6) - 72 + 360\left(\frac{1}{k} + \frac{1}{l}\right) . \tag{20.44}$$

(20.42) is recovered by taking $k = 24, n = 2l - 16$. Further Diophantine conditions, analogous to (20.43), reduce the solutions to $k, l = 8, 9, 10, 12, 15, 18, 24, 35$. The corresponding Lie algebras form the magic triangle, fig. 1.1.

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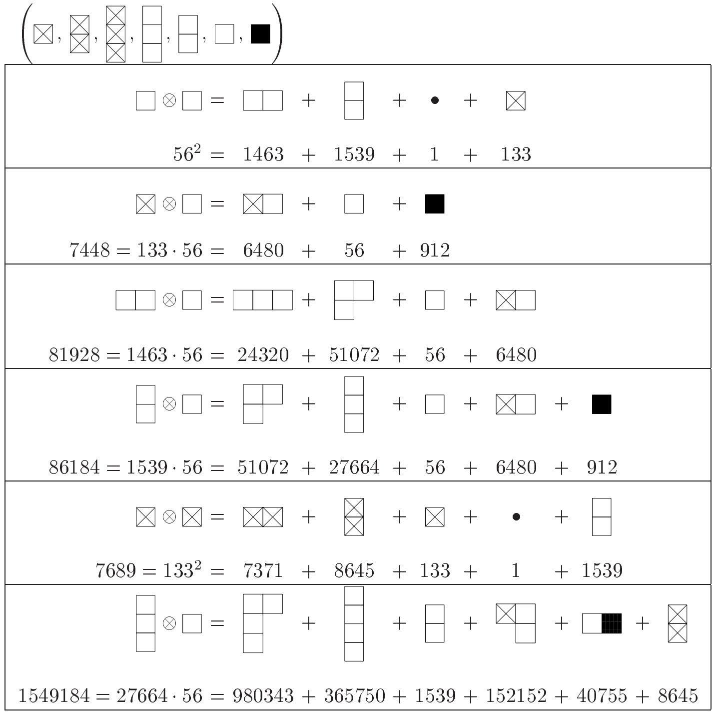

Table 20.5 TABLE FROM END OF F4FAMILY CHAPTER. NEEDS EDITING!

Chapter Twenty One

Exceptional magic

The study of invariance algebras as pursued in chapter 16 to 20 might appear to be a rather haphazard affair. Given a set of primitives, one gets some Diophantine equations, constructs the family of invariance algebras and moves onto the next set of primitives. However, a closer scrutiny of the Diophantine conditions leads to a surprise: most of the Diophantine equations are special cases of one and the same Diophantine equation, and they magically arrange all exceptional families into a single triangular pattern which we shall call the Magic Triangle.

21.1 MAGIC TRIANGLE

Our construction of invariance algebras has generated a series of Diophantine conditions which we now summarize. The adjoint rep conditions are:

$$\begin{array}{ll} F_4 \text{ family} & N = 3n - 36 + \dfrac{360}{n+10} \\ E_6 \text{ family} & N = 4n - 40 + \dfrac{360}{n+9} \\ E_7 \text{ family} & N = 3n - 45 + \dfrac{360}{n/2+8} \\ E_8 \text{ family} & N = 10m - 122 + \dfrac{360}{m}\,. \end{array} \tag{21.1}$$

There is a striking similarity between the conditions for different families. If we define

$$\begin{array}{ll} F_4 \text{ family} & m = n + 10 \\ E_6 \text{ family} & m = n + 9 \\ E_7 \text{ family} & m = n/2 + 8\,, \end{array} \tag{21.2}$$

we can parametrize all the solutions of the above Diophantine conditions with a single integer m, see table 21.1. The Clebsch-Gordan series for $A \otimes V$ Kronecker products also show a striking similarity. The characteristic equations (17.11), (18.28), (**??**) and (**??**) are the one and the same equation

$$(\mathbf{Q} - 1)\left(\mathbf{Q} + \frac{6}{m}\right) P_r = 0\,. \tag{21.3}$$

Here P_r removes the defining and $\otimes V^2$ subspaces, and we have rescaled the E_8 operator $\mathbf{Q}$ (17.11) by factor 2. (Role of the $\mathbf{Q}$ operator is only to distinguish between two subspaces - we are free to rescale it, as we wish).

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m	8	9	10	12	15	18	20	24	30	36	40	$\cdots$	360
F_4			0	0	3	8	.	21	.	52	.	$\cdots$	.
E_6		0	0	2	8	16	.	35	36	78	.	$\cdots$	.
E_7	0	1	3	9	21	35	.	66	99	133	.	$\cdots$	.
E_8	3	8	14	28	52	78	.	133	190	248	.	$\cdots$	.

Table 21.1 All solutions of Diophantine conditions (21.1) not eliminated by other Diophantine conditions of chapter 16 through 19; those are marked by "·".

In the dimensions of the associated reps, eigenvalue $6/m$ introduces a new Diophantine denominator $m + 6$. For example, from (17.19), table 18.4, (**??**) and (**??**), the highest dimensional rep in $V \otimes A$ has dimension (in terms of parametrization (21.2)):

$$\begin{array}{ll} F_4 \text{ family} & 3(m+6)^2 - 156(m+6) + 2673 - \dfrac{15120}{m+6} \\ E_6 \text{ family} & 4(m+6)^2 - 188(m+6) + 2928 - \dfrac{15120}{m+6} \\ E_7 \text{ family} & 2\left\{6(m+6)^2 - 246(m+6) + 3348 - \dfrac{15120}{m+6}\right\} \\ E_8 \text{ family} & 50m^2 - 1485m + 19350 + \dfrac{27 \cdot 360}{m} - \dfrac{11 \cdot 15120}{m+6}\,. \end{array} \tag{21.4}$$

These Diophantine conditions eliminate most of the spurious solutions of (21.1); only the $m = 30, 60, 90$ and 120 spurious solutions survive but are in turn eliminated by further conditions. For the E_8 family, $V \otimes V = V \otimes A = A \otimes A$ (the defining rep is the adjoint rep), hence, the Diophantine condition (21.4) includes both $1/m$ and $1/(m+6)$ terms. Not only can the four Diophantine conditions (21.1) be parametrized by a single integer m; the list of solutions table 21.1 turns out to be symmetric under the flip across the diagonal. F_4 solutions are the same as those in the $m = 15$ column, and so on. This suggests that the rows be parametrized by an integer ℓ, in a fashion symmetric to the column parametrization by m. Indeed, the requirement of $m \leftrightarrow \ell$ symmetry leads to a unique expression which contains the four Diophantine conditions (21.1) as special cases:

$$N = \frac{(\ell - 6)(m - 6)}{3} - 72 + \frac{360}{\ell} + \frac{360}{m} \tag{21.5}$$

We take $m = 8, 9, 10, 12, 15, 18, 24, 30$ and 36 as all the solutions allowed in table 21.1. By symmetry ℓ takes the same values. All the solutions fill up the *Magic Triangle*, table 21.1. Within each entry, the number in the upper left corner is N, the dimension of the corresponding Lie algebra, and the number in the lower left corner is n, the dimension of the defining rep. The expressions for n for the top four rows are guesses. The triangle is called magic, partly because we arrived at it by magic, and partly because it contains Freudenthal's Magic Square [72], marked by the dotted line in table 21.1.

						0 0	3 $\mathbf{A}_1$ 2
					0 0	1 $\mathbf{U}(1)$ 1	8 $\mathbf{A}_2$ 3
				0 0	0 1	3 $\mathbf{A}_1$ 3	14 $\mathbf{G}_2$ 7
			0 0	0 1	2 $2\mathbf{U}(1)$ 2	9 $3\mathbf{A}_1$ 4	28 $\mathbf{D}_4$ 8
		0 0	0 2	3 $\mathbf{A}_1$ 5	8 $\mathbf{A}_2$ 8	21 $\mathbf{C}_3$ 14	52 $\mathbf{F}_4$ 26
	0 0	0 1	2 $2\mathbf{U}(1)$ 3	8 $\mathbf{A}_2$ 6	16 $2\mathbf{A}_2$ 9	35 $\mathbf{A}_5$ 15	78 $\mathbf{E}_6$ 27
0 0	1 $\mathbf{U}(1)$ 2	3 $\mathbf{A}_1$ 4	9 $3\mathbf{A}_1$ 8	21 $\mathbf{C}_3$ 14	35 $\mathbf{A}_5$ 20	66 $\mathbf{D}_6$ 32	133 $\mathbf{E}_7$ 56
3 $\mathbf{A}_1$ 3	8 $\mathbf{A}_2$ 8	14 $\mathbf{G}_2$ 14	28 $\mathbf{D}_4$ 28	52 $\mathbf{F}_4$ 52	78 $\mathbf{E}_6$ 78	133 $\mathbf{E}_7$ 133	248 $\mathbf{E}_8$ 248

Table 21.2 *Magic triangle.* All exceptional Lie groups defining and adjoint reps form an array of the solutions of the Diophantine condition (21.5). Within each entry the number in the upper left corner is N, the dimension of the corresponding Lie algebra, and the number in the lower left corner is n, the dimension of the defining rep.

21.2 LANDSBERG-MANIVEL CONSTRUCTION

Inspired by conjectures of Deligne (see sect. 17.5.2), in a series of papers J. M. Landsberg and L. Manivel [184, 185, 186, 187] apply a wide spectrum of methods, from Cartan subalgebras to trialities and projective geometries, in order not only to interpret the Freudhental Magic Square, but also to extend it. They arrive at some of the formulas derived here, including the column of non-reductive algebras in table 17.2. They deduce the formula (21.5) conjecturede above from the Vogel's [202] "universal Lie algebra" dimension formula (proposition 3.2 of ref. [186]), and interpret m, ℓ as $m = 3(a+4)$, $\ell = 3(b+4)$, where $a, b = 0, 1, 2, 4, 6, 8$ are the dimensions of division algebras (see sect. 16.3) used in their construction. For $m \geq 12$ this agrees with the Freudhental Magic Square, but for $m \leq 10$ the corresponding "division algebras" would need to be of dimensions $a = -2/3, -1, -4/3, -2$, a somewhat unnatural state of affairs.

21.3 EPILOGUE

> Because something is happening here
> But you don't know what it is
> Do you, Mister Jones?
> Bob Dylan: "Ballad of a thin man"

"I read your book. It is long and it seems original, but - why? Why did you do this?" you might well ask. OK, here is an answer.

Looking back, almost everything I have done as work which I was paid to do will probably be of no lasting interest - while the things that I did on the side, for my own pleasure, have in the long run turned out to be the insights worth living for. One such sidetrack has to do with a conjecture of finiteness of gauge theories, which, by its own twisted logic, led to this sidetrack, birdtracks and exceptional Lie algebras.

I started out as a condensed matter experimentalist at MIT, and as such I was brought to Cornell as a Xerox fellow. I once went down into the bowels of Clark Hell, where a professor with an army of people was slaving away in dark cubicles, and I promptly decided to join instead the field theorists who owned a beautiful rooftop view of the Ithaca hills and Ithaca skies[1]. One fateful day Toichiro Kinoshita came up with a Feynman integral and asked me whether I could evaluate it for him. No sweat, I worked for a while and not only did I integrate it, but also I gave a formula for all Feynman integrals of that topology. It was only a bait. He came up with the next integral on which my general method failed miserably. Then he came with the next integral, and it was like Vietnam - there was no way of getting out of it. I was spending nights developing algebraic languages disguised as editor macros so that synchrotron experimentalists would let me use their computer; we were flying in small planes to Brookhaven, carrying suitcases of computer punch-cards; and by four years later we had completed what at that time was the most complicated and the most expensive calculation ever carried out on a computer, and the answer

[1] Winter 1996: Doug Osheroff who arrived a year earlier, stayed on.

was [35]:

$$\frac{1}{2}(g-2) = \frac{1}{2}\frac{\alpha}{\pi} - 0.32848\left(\frac{\alpha}{\pi}\right)^2 + (1.183 \pm 0.011)\left(\frac{\alpha}{\pi}\right)^3 .$$

At the very end, I dreamed that I was a digit toward the end of the long string of digits that we had calculated for the electron magnetic moment, and that I died by being dropped as an insignificant digit. I was ready to move on.

Among my friends at Cornell were two called Feigenbaum. The first one moved to a factory town to do union organizing and reached brief national fame when the Mafia bombed his house. The other one was amazingly fast in solving New York Times crossword puzzles, but he published nothing. Hans Bethe dispatched him to Blackhole, Virginia, where he languished publishing nothing until Peter Carruthers rescued him and took him to Los Alamos on the risky presumption that the man seemed very smart. In contrast to these good-for-nothings, I was advertised as the best thing since Roman Jackiw and sent off to Stanford, Princeton and Oxford with a mission to solve the QCD quark confinement problem.

So, what did I do? I found myself in California, my reading of Nietzsche came to an abrupt halt, to be replaced by volleyball, bicycling and scortatory love. I wrote dutifully a series of papers allegedly curing the infrared ills of QCD, and - well, we never did solve the quark confinement problem, not to this day, not in my book, at least.

But one day, terror struck; I was invited to Caltech to give a talk. I could go to any other place and say that Kinoshita and I have computed thousands of diagrams and that the answer is, well, the answer is:

$$+(0.92 \pm 0.02)\left(\frac{\alpha}{\pi}\right)^3 .$$

But in front of Feynman? He is going to ask me why "+" and not "-"? Why do 100 diagrams yield a number of order of unity, and not 10 or 100 or any other number? It might be the most precise agreement between a fundamental theory and experiment in all of physics - but what does it mean?

Now, you probably do not know how stupid the quantum field theory is in practice. What is done (or at least was done, before the field theorists left this planet for pastures beyond the Planck length) is:

1) start with something eminently sensible (electron magnetic moment; positronium)

2) expand this into combinatorially many Feynman diagrams, each an integral in many dimensions with integrand with thousands of terms, each integral UV divergent, IR divergent, and meaningless, as its value depends on the choice of gauge

3) integrate by Monte Carlo methods in 10-20 dimensions this integral with dreadfully oscillatory integrand, and with no hint of what the answer should be; in our case ± 10 to ± 100 was a typical range

4) add up hundreds of such apparently random contributions and get

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$$+(0.92 \pm 0.02)\left(\frac{\alpha}{\pi}\right)^3 .$$

So, for the fear of Feynman I went into deep trance and after a month came up with this:

If gauge invariance of QED guarantees that all UV and IR divergences cancel, why not also the finite parts?

And indeed; when the diagrams that we had computed are grouped into gauge invariant subsets, a rather surprising thing happens [38]; while the finite part of each Feynman diagram is of order of 10 to 100, every subset adds up to approximately

$$\frac{1}{2}\left(\frac{\alpha}{\pi}\right)^n .$$

If you take this numerical observation seriously, the "zeroth" order approximation to the electron magnetic moment is given by

$$\frac{1}{2}(g-2) = \frac{1}{2}\frac{\alpha}{\pi}\frac{1}{\left(1-\left(\frac{\alpha}{\pi}\right)^2\right)^2} + \text{“corrections"}.$$

Now, this is a great heresy - my colleagues will tell you that Dyson has shown that the perturbation expansion is an asymptotic series, in the sense that the nth order contribution should be exploding combinatorially

$$\frac{1}{2}(g-2) \approx \cdots + n^n\left(\frac{\alpha}{\pi}\right)^n + \cdots ,$$

and not growing slowly like my estimate

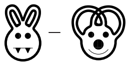
$$\frac{1}{2}(g-2) \approx \cdots + n\left(\frac{\alpha}{\pi}\right)^n + \cdots .$$

But do not take them too seriously - very few of them have ever computed anything. For me, these unreasonably effective cancellations are a tantalizing hint that something deep, deeper than anything what we know today, lurks in the gauge invariance of quantum field theories.

I should not have bothered. I was fated to arrive from SLAC to Caltech precisely five days after the discovery of the J/ψ particle. I had to give an impromptu irrelevant talk about what would the total e^+e^- cross-section had looked like if J/ψ were a heavy vector boson, and had only 5 minutes for my conjecture about the finiteness of gauge theories. Feynman liked it and gave me sage and thoroughly irrelevant advice. I kept looking for a simpler gauge theory in which I could compute many orders in perturbation theory and check the conjecture. We learned how to count Feynman diagrams [39]. I invented the planar field theory [40] whose perturbation expansion is convergent, but did not know how to combine this with gauge invariance. I formulated the theory of the group weights of Feynman diagrams in non-Abelian gauge theories [36] (chapters 3–15 of this monograph) but did not find a relative of local gauge invariance there, either. By marrying Poincaré to Feynman we found a new perturbative expansion [46] which appears more compact than the standard Feynman diagram perturbation theory. No dice. To this day I still do not know how to prove or disprove the conjecture.

Los Angeles is no place for a car hater, so I eloped for Institute for Advanced Study, Princeton. The Institute is a quiet pretty place at the edge of the woods. The mathematician who lived in my apartment before me had been taken away by men in white coats because he never spoke, he only grunted. They found the apartment furnished by a large number of dictionaries, and nothing else. I loved being there. During the day I was solving the quark confinement problem (Stephen Adler got me into some cockeyed quaternionic calculation), but the nights were mine. I still remember the bird song, the pink of the breaking dawn, and me ecstatically pursuing the next tangent:

QCD quarks are supposed to come in three colors. This requires evaluation of SU(3) group theoretic factors, something anyone can do. In the spirit of Teutonic completeness, I wanted to check all possible cases; what would happen if the nucleon consisted of 4 quarks, doodling

 $= n(n^2 - 1)$,

and so on, and so forth? In no time, and totally unexpectedly, all exceptional Lie groups arose, not as Diophantine conditions on Cartan lattices, but on the same geometrical footing as the classical invariance groups of quadratic norms, $SO(n)$, $SU(n)$ and $Sp(n)$.

If I were to give myself a prize - I am not thinking of anything big, but of something commensurate with what I have accomplished, let us say a week's vacation in Lalandia - I would have given myself a prize for the Magic Triangle, table 21.1, where all exceptional Lie groups emerge in one big family. I like this, because it is one of those magic things that one discovers for no apparent reason whatsoever. Now, I am a fool who, even though he has put more effort in this project than any other, has only completed a write up here and now, in the pages that you have just leafed through. This, first because nobody wants to hear about it, second because I have no idea about how to derive all rows of the Magic Triangle in one go, and then, the truth: I got sidetracked by the next equally frivolous side diversion.

In spring 1976 Mitchell Feigenbaum came to visit from Los Alamos, having published even less than before. He gave a seminar, but nobody understood a word. Starting point was a parabola, then things got incredibly complicated, and at the end it turned out that the theory might be applicable to fluctuations of forest moth populations. However, Mitchell and I were driven by a secret agenda - the thing was robust, you could make it very imperfect, and a universal superstructure would survive the imperfections. In other words, just what you need to build a brain - all parts imperfect, and the thing functions anyway. The grand scheme boiled down to one equation [64],

$$g(x) = \alpha g(g(x/\alpha)) \,,$$

and I went off to the math library to look it up. The Institute has an excellent math library, but I did not find it. As a matter of fact, we never found it to this very day - it had never been written down before.

As you would expect, nobody wanted to hear about it, either. To be fair, I remember the total of four who did: Freeman Dyson, John Milnor, Bill Thurston, and Blott. Blott is a wonderful San Franciscan whom I love even more dearly than Dyson; the rest you should know.

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By that time I was already deep in trouble - once I learned that chaos is generic for generic Hamiltonian flows, I lost faith in doing field theory by pretending that it is a bunch of harmonic oscillators, with interactions accounted for as perturbative corrections. This picture is simply wrong - strongly coupled field theories (hydrodynamics, QCD, gravity) are nothing like that [47]. So they excommunicated me from the ranks of high energy theorists, and now I am in charge of a project. We are working out quantum chaos. The truth is, this is what I set out to do in 1978 - replace the path integral with a fractal set of semi-classical orbits, so I feel that I am closer to figuring out the quark confinement than ever before.

"OK, OK - everybody has a story," you say "But on page 190 you claim to have derived your Magic Triangle [36, 37, 41] already in late 1970's. What took you so long?"

Some water has flown under the bridge in the meantime, and diagrammatic methods have since become the notation of choice for a select group of group theory practitioners. However, nobody, but truly nobody showed a glimmer of interest in the exceptional Lie algebra parts of this work, so there was no pressure to publish it before completing it: by completing it I mean finding the algorithms that would reduce any bubble diagram to a number for any semi-simple Lie algebra. This monograph accomplishes the task for G_2, but for F_4, E_6, E_7 and E_8 this is still an open problem. This, perhaps, is only matter of algebra (all computations in this monograph were done by hand, mostly on trains and in airports), but the truly frustrating unanswered question is:

Where does the Magic Triangle come from? Why is it symmetric across the diagonal? The Freudenthal-Tits construction of the Magic Square in terms of octonionic matrices is the best answer so far, but it is not a natural answer from the invariance groups perspective. Something is happening here, but I don't know what it is. A Mother of All Lie Algebras, some complex function which yields the Magic Triangle for a set of integer values? This, it seems, requires an idea. Une idée, c'est déjà quelque chose.

And then there is a practical issue of unorthodox notation: transferring birdtracks from hand drawings to LaTeX took another 21 years. In this I was rescued by Henriette Elvang who mastered the art of birdtracking on her own, in her Master's Thesis, and introduced me to Anders Johansen, Copenhagen University undergraduate, who then undertook drawing some 4,000 birdtracks needed to complete this manuscript, of elegance far outstriping that of the old masters.

Remain brave.

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Chapter Twenty Two

Magic negative dimensions

22.1 E_7 AND $SO(4)$

22.2 E_6 AND $SU(3)$

Appendix A

Recursive decomposition

This appendix deals with practicalities of computing projection operator eigenvalues, and is best skipped unless you need to carry out such calculation.

Let P stand for a projection onto a subspace or the entire space (in which case $P = \mathbf{1}$). Assume that the subspace has already been reduced into m irreducible subspaces and a reminder

$$P = \sum_{\gamma=1}^{m} P_\gamma + P_r \,. \tag{A.1}$$

Now adjoin a new invariant matrix Q to the set of invariants. By assumption, Q does not reduce further the $\gamma = 1, 2, \ldots, m$ subspaces, *ie.* has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$

$$QP_\gamma = \lambda_\gamma P_\gamma \qquad \text{(no sum)}\,, \tag{A.2}$$

on the γth subspace. We construct an invariant, matrix $\hat{Q}$, restricted to the remaining (as yet not decomposed) subspace by

$$\hat{Q} := P_r Q P_r = PQP - \sum_{\gamma=1}^{m} \lambda_\gamma P_\gamma \,. \tag{A.3}$$

As P_r is a finite dimensional subspace, $\hat{Q}$ satisfies a *minimal* characteristic equation of order $n \geq 2$

$$\sum_{k=0}^{n} a_k \hat{Q}^k = \prod_{\alpha=m+1}^{m+n} (\hat{Q} - \lambda_\alpha P_r) = 0 \,, \tag{A.4}$$

with the corresponding projection operators (3.45).

$$P_\alpha = \prod_{\beta \neq \alpha} \frac{\hat{Q} - \lambda_\beta}{\lambda_\alpha - \lambda_\beta} P_r \,, \qquad \alpha = \{m+1, \ldots, m+n\} \,. \tag{A.5}$$

"Minimal" in the above means that we drop repeated roots, so all eigenvalues are distinct. $\hat{Q}$ is an awkward object in computations, so we reexpress the projection operator, in terms of Q, as follows.

Define first the polynomial, obtained by deleting the $(\hat{Q} - \lambda_\alpha)$ factor from (A.4)

$$\prod_{\beta \neq \alpha} (x - \lambda_\beta) = \sum_{k=0}^{n-1} b_k x^k \,, \qquad \alpha, \beta = m+1, \ldots m+n \,, \tag{A.6}$$

where the expansion coefficient $b_k = b_k^{(\alpha)}$ depends on the choice of the subspace α. Substituting $P_r = P - \sum_{\alpha=1}^{m} P_\alpha$ and using the orthonomality of P_α, we obtain

an alternative formula for the projection operators

$$P_\alpha = \frac{1}{\sum b_k \lambda_\alpha^k} \sum_{k=0}^{n-1} b_k \left\{ (PQ)^k - \sum_{\gamma=1}^{m} \lambda_\alpha^k P_\gamma \right\} P \,, \tag{A.7}$$

and dimensions

$$d_\alpha = \operatorname{tr} P_\alpha = \frac{1}{\sum b_k \lambda_\alpha^k} \sum_{k=0}^{n-1} b_k \left\{ \operatorname{tr} (PQ)^k - \sum_{\gamma=1}^{m} \lambda_\gamma^k d_\gamma \right\} \,. \tag{A.8}$$

The utility of this formula lies in the fact that once the polynomial (A.6) is given, the only new data it requires, are the traces $\operatorname{tr} (PQ)^k$, and those are simpler to evaluate than $\operatorname{tr} \hat{Q}^k$.

Appendix B

Properties of Young projections

(H. Elvang and P. Cvitanović)

In this appendix we prove the properties of the Young projection operators, stated in sect. 9.4.

B.1 UNIQUENESS OF Young projection operators

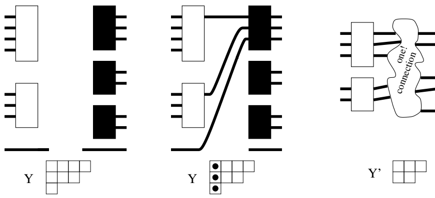
We now show that the Young projection operator P_{Y} is well-defined by proving the existence and uniqueness (up to sign) of a non-vanishing connection between the symmetrizers and antisymmetrizers in P_{Y}.

The proof is induction over the number of columns t in the Young diagram Y. For $t = 1$ the Young projection operator consists of one antisymmetrizer of length s and s symmetrizers of length 1, and clearly the connection can only be made in one way, up to an overall sign.

Assume the result to be valid for Young projection operators derived from Young diagrams with $t-1$ columns. Let Y be a Young diagram with t columns. The lines from A_1 in P_{Y} must connect to different symmetrizers for the connection to be non-zero. Since there are exactly $|\mathrm{A}_1|$ symmetrizers in P_{Y}, this can be done in essentially one way, since which line goes to which symmetrizer is only a matter of an overall sign, and where a line enters a symmetrizer is irrelevant due to (6.8).

After having connected A_1, connecting the symmetry operators in the rest of P_{Y} is the problem of connecting symmetrizers to antisymmetrizers in the Young projection operator $P_{\mathrm{Y}'}$, where Y' is the Young diagram obtained from Y by slicing off the first column. Thus, Y' has $k-1$ columns, so by the induction hypothesis, the rest of the symmetry operators in P_{Y} can be connected in exactly one non-vanishing way (up to sign).

The principles are illustrated below:

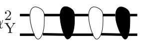
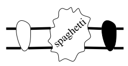


(B.1)

B.2 NORMALIZATION

We now derive the formula for the normalization factor α_{Y}, such that the Young projection operators are idempotent, $P_{\mathrm{Y}}^2 = P_{\mathrm{Y}}$. By the normalization of the symmetry operators, Young projection operators derived from fully symmetrical or antisymmetrical Young tableaux, will be idempotent with $\alpha_{\mathrm{Y}} = 1$.

P_{Y}^2 is simply P_{Y} connected to P_{Y}, hence, it may be viewed as a set of *outer* symmetry operators connected by a set of *inner* symmetry operators. Expanding all the inner symmetrizers and using the uniqueness of the non-zero connection between the symmetrizers and antisymmetrizers of the Young projection operator, we find that each term in the expansion is either 0 or a version of P_{Y}. In fact, the number of non-zero terms — denote it $\|\mathrm{Y}\|$ — is just the number $|\mathrm{Y}|$, defined in sect. 9.4. For a Young diagram with s rows and t columns, there will be a factor of $\frac{1}{|\mathrm{S}_i|}$ $(\frac{1}{|\mathrm{A}_i|})$ for expansion of each inner (anti)symmetrizer, thus we find

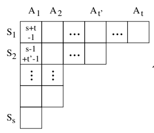
$$P_{\mathrm{Y}}^2 = \alpha_{\mathrm{Y}}^2$$  $$= \frac{\alpha_{\mathrm{Y}}^2}{\prod_{i=1}^{s}|\mathrm{S}_i|!\prod_{j=1}^{t}|\mathrm{A}_j|!}\sum_{\mathrm{mess}}$$ 

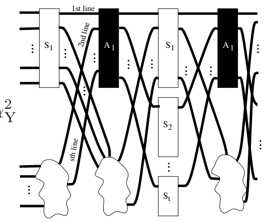
$$= \alpha_{\mathrm{Y}}\frac{|\mathrm{Y}|}{\prod_{i=1}^{s}|\mathrm{S}_i|!\prod_{j=1}^{t}|\mathrm{A}_j|!}P_{\mathrm{Y}}. \tag{B.2}$$

Idempotency is then achieved by taking

$$\alpha_{\mathrm{Y}} = \frac{\prod_{i=1}^{s}|\mathrm{S}_i|!\prod_{j=1}^{t}|\mathrm{A}_j|!}{|\mathrm{Y}|}. \tag{B.3}$$

Let Y be a Young tableau with $|\mathrm{A}_1| = s$, $|\mathrm{S}_1| = t$, $|\mathrm{S}_2| = t'$ *etc.* We count in how many ways the lines, entering the inner A_1, pass through it to yield non-zero connections. We refer to

$$\mathrm{Y} =$$ 

$$\rightsquigarrow P_{\mathrm{Y}}^2 = \alpha_{\mathrm{Y}}^2$$ 

(B.4)

in the following. For each of the inner symmetrizers there must be exactly one from A_1. The first line can pass through A_1 in s ways, and without loss of generality we may take it to pass straight through, connecting to S_1 where it can pass through in t ways. Thus for the first line, there were $s+t-1$ allowed roads through the inner symmetry operators. The second line may now pass through A_1 in $s-1$ ways, and we can take it to pass straight through to S_2, where it has t' possibilities. Thus, we have found $(s-1)+t'-1$ options for the second line. With a similar reasoning we find $(s-2)+t''-1$ allowed ways for the third line, *etc.*

Let w_{Y} be the number of ways of passing the m lines entering A_1 through the inner symmetry operators. w_{Y} is then the product of the numbers found above,

$w_{\rm Y} = (s+t-1)(s-1+t'-1)(s-2+t''-1)\cdots$. Note that when calculating $|{\rm Y}|$, the product of the numbers in the first column of the Young diagram is $w_{\rm Y}$.

We show $\|{\rm Y}\| = |{\rm Y}|$ by induction on the number of columns t in the Young diagram Y.

For a single column Young diagram, $|{\rm Y}| = |{\rm A}_1|!$, and the number of non-zero ways to connect the ${\rm A}_1$ symmetrizers to ${\rm A}_1$ in $P_{\rm Y}$ is $|{\rm A}_1|!$, hence, $\|{\rm Y}\| = |{\rm Y}|$ for $t = 1$.

Assume that $\|{\rm Z}\| = |{\rm Z}|$ for any Young diagram Z with $t-1$ columns. Let Y be a Young diagram with t columns, and let ${\rm Y}'$ be the Young diagram obtained form Y by removal of the first column. $w_{\rm Y}$ is the number of ways, the lines, entering the first inner antisymmetrizer in $P_{\rm Y}^2$, are allowed to pass through the inner symmetry operators. Finding the number of allowed paths for the rest of the lines, is the problem of finding the number of allowed paths through the inner symmetry operators of $P_{{\rm Y}'}^2$, which is $\|{\rm Y}'\| = |{\rm Y}'|$. Now we have $\|{\rm Y}\| = \|{\rm Y}'\| w_{\rm Y} = |{\rm Y}'| w_{\rm Y} = |{\rm Y}|$.

B.3 ORTHOGONALITY

If Y and Z denote Young tableaux derived from the same Young diagram, then $P_{\rm Y} P_{\rm Z} = P_{\rm Z} P_{\rm Y} = \delta_{Y,Z} P_{\rm Y}$, since there is a non-trivial permutation of the lines connecting the symmetry operators of Y with those of Z, and by uniqueness of the non-zero connection, the result is either $P^2 = P$ or 0.

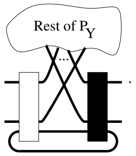
Next, consider two *differently shaped* Young diagrams Y and Z with the same number of boxes. Since at least one column must be bigger in (say) Y than in Z, and the p lines from the corresponding antisymmetrizer must connect to different symmetrizers, it is not possible to make a non-zero connection between the anti-symmetry operators of $P_{\rm Y}$ to the symmetrizers in $P_{\rm Z}$, and hence, $P_{\rm Y} P_{\rm Z} = 0$. By a similar argument, $P_{\rm Z} P_{\rm Y} = 0$.

B.4 THE DIMENSION FORMULA

The dimensions of the irreducible reps can be calculated recursively from the Young projection operators. Here is the recipe:

Let Y be a Young diagram and ${\rm Y}'$ the Young diagram obtained from Y by removal of the right-most box in the last row. Draw the Young projection operators corresponding to Y and ${\rm Y}'$, and note that if we trace the last line of $P_{\rm Y}$, we obtain $P_{{\rm Y}'}$ multiplied by a factor.

Quite generally, this contraction will look like



(B.5)

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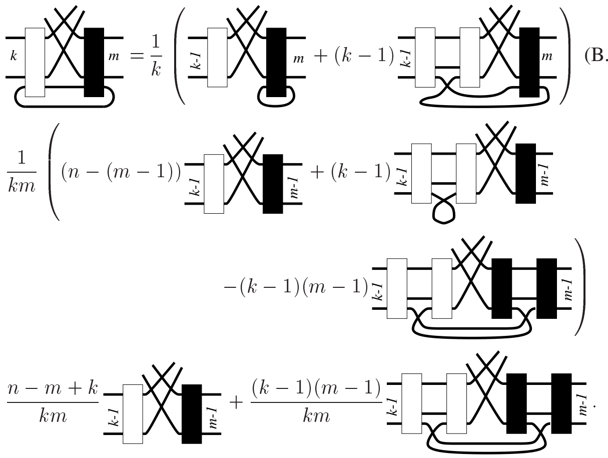
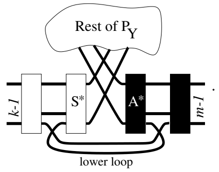
Using (6.10) and (6.19), we have



(B.6)

Inserting (B.6) into (B.5), we see that the first term is proportional to the projection $P_{{\rm Y}'}$. The second term vanishes:



(B.7)

The lines, going into ${\rm S}^*$, come from antisymmetrizers in the rest of the $P_{\rm Y}$-diagram. One of these lines, from ${\rm A}_a$, say, must pass from ${\rm S}^*$ through the lower loop to ${\rm A}^*$ and from ${\rm A}^*$ connect to one of the symmetrizers, say ${\rm S}_S$, in the rest of the $P_{\rm Y}$-diagram. But due to the construction of the connection between symmetrizers and antisymmetrizers in a Young projection operator, a line is already connecting ${\rm S}_s$ to ${\rm A}_a$. Hence, the diagram vanishes.

The dimensionality formula follows by induction on the number of boxes in the Young diagrams, with the dimension of a single box Young diagram being n. Let Y be a Young diagram with p boxes. We assume that the dimensionality formula is valid for any Young diagram with $p-1$ boxes. With $P_{{\rm Y}'}$ obtained from $P_{\rm Y}$ as above, we have (using (B.6) and writing $D_{\rm Y}$ for the birdtrack diagram of $P_{\rm Y}$):

$$\dim P_{\rm Y} = \alpha_{\rm Y} {\rm tr}\, {\rm D}_{\rm Y} = \frac{\rm n-m+k}{\rm km} \alpha_{\rm Y} {\rm tr}\, {\rm D}_{{\rm Y}'} \tag{B.8}$$

$$= (n-m+k) \alpha_{{\rm Y}'} \frac{|{\rm Y}'|}{|{\rm Y}|} {\rm tr}\, {\rm D}_{{\rm Y}'} \tag{B.9}$$

$$= (n-m+k) \frac{f_{{\rm Y}'}}{|{\rm Y}|} = \frac{f_{\rm Y}}{|{\rm Y}|}\,. \tag{B.10}$$

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This completes the proof of the dimensionality formula (9.25).

B.5 LITERATURE

- This introduction to the Young tableaux is based on Lichtenberg [105], Hamermesh [80] and van der Waerden [154].
- The rules for reduction of direct products: See Lichtenberg [105]. The rules are stated here as in (Elvang 1999).
- The method of constructing the Young projection operators, directly from the Young tableaux, is described in van der Waerden [154], who ascribes the idea to von Neumann. See also Kennedy slides [96].
- Alternative labeling of Young diagrams: Fischler [67].

$$(a_1 a_2 \ldots a_{r-1} Z) \to (a_1 a_2 \ldots a_k 00 \ldots) \ . \tag{B.11}$$

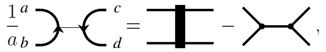
Appendix C

G_2 calculations

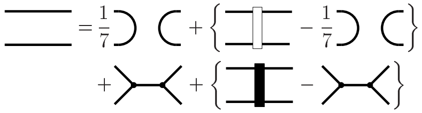
C.1 EVALUATION RULES FOR G_2

The G_2 invariance algebra is derived in chapter 16. The evaluation rules are:
Adjoint rep A:

$$= \frac{1}{2}\left(\delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd}\right) - f_{abe}f_{ecd}$$

 (C.1)

$V \otimes V$ decomposition:

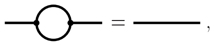
Projector: 

Dimension: $n^2 = 1 + +7 + 14$

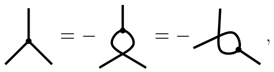
Dynkin index: $l^{-1} = + + 1$ (C.2)

f_{abc} algebra is defined by

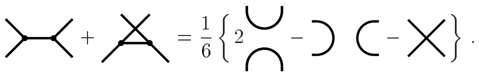
normalization

 (C.3)

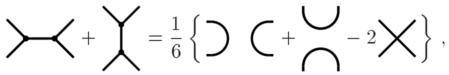
total antisymmetry

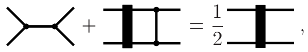
 (C.4)

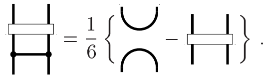
and the alternativity relation

 (C.5)

Other forms of the alternativity relation are

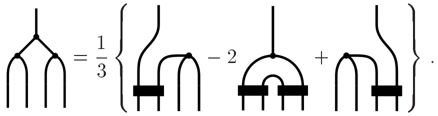
 (C.6)

 (C.7)

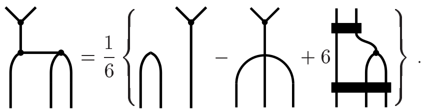
 (C.8)

From the above three defining relations follow all other identities:

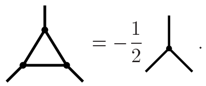
Reduction identity provides the algorithm for evaluating any color weight:

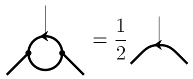
 (C.9)

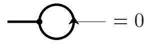
Another form of the reduction identity is

 (C.10)

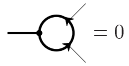
Sundry relations:

 (C.11)

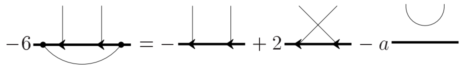
 (C.12)



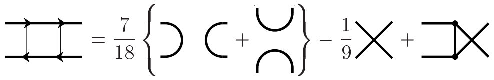
(C.13)



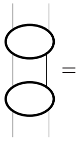
(C.14)



(C.15)



(C.16)



(C.17)

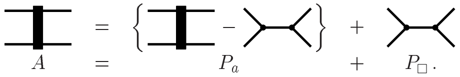
C.2 G_2, FURTHER CALCULATIONS

Some formulas (not to be included into the manuscript) for symmetric reps (-9) :

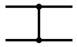
$$d_\pm = \frac{(n+2)}{4}\left\{n-1\pm(n^2-3n-30)\sqrt{\frac{n+2}{n^3+\dots}}\right\}. \tag{C.18}$$

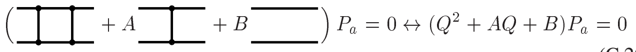
The $n^2-3n-30$ seem to be off by an extra factor of 4?

C.2.1 G_2 antisymmetric $V\otimes V$ subspace

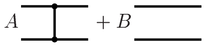


(C.19)

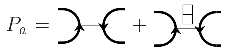
The P_a is split by primitiveness. $Q =$ 



(C.20)

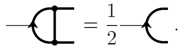
The A  are the only trees on P_a subspace.

Invariance condition: Know that it must contain the adjoint rep:



(C.21)

and that the adjoint rep has eigenvalue $\frac{1}{2}$:



(C.22)

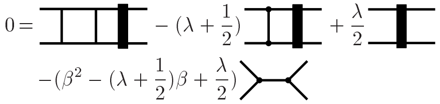
The remaining eigenvalue λ needs to be fixed. The projectors are

$$P_{\square} = \frac{Q-1/2}{\lambda-1/2}P_a,\; P_A = \frac{Q-\lambda}{1/2-\lambda}P_a\,, \tag{C.23}$$

and the characteristic equation is

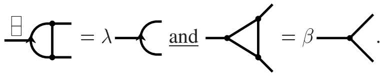
$$(Q^2-(\lambda+1/2)Q+\lambda/2)P_a = 0\,. \tag{C.24}$$

This eliminates A, B above in favor of single parameter λ. However, there are 2 parameters. Expanding P_λ get

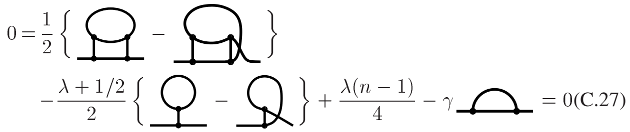


(C.25)

$$\beta^2-(\lambda+\frac{1}{2})\beta+\frac{\lambda}{2} = (\beta-\lambda)(\beta-\frac{1}{2}) \equiv \gamma\,. \tag{C.26}$$

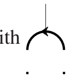
So we need to fix 

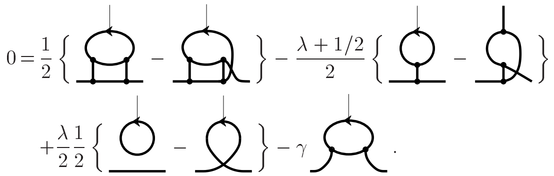
Trace, contract with ——— from above, get



(C.27)

$$\boxed{1-\beta-(\lambda+\tfrac{1}{2})+\tfrac{\lambda}{2}(n-1)-2\gamma = 0}\,. \tag{C.28}$$

Trace with 

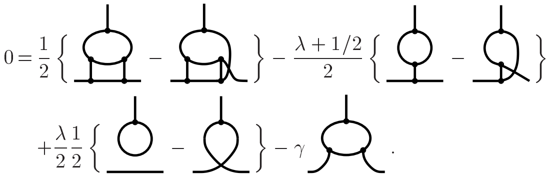


(C.29)

The second term in the first bracket and the first term in the second bracket both equal 0 by symmetry.

$$\frac{1}{4}-(\lambda+\frac{1}{2})(-\frac{1}{2})+\frac{\lambda}{2}-\frac{\gamma}{2} \Rightarrow \boxed{\lambda+\tfrac{1}{2}=\gamma}\,. \tag{C.30}$$

Trace with 



(C.31)

The second term in the first bracket equals 0 by symmetry.

$$\beta^2 - (\lambda + \frac{1}{2})(1-\beta) + \frac{\lambda}{2} - 2\beta\gamma = 0\,. \tag{C.32}$$

Replace γ by (C.30):

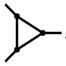
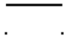
$$\beta^2 - (\lambda + \frac{1}{2})(1-\beta) + \frac{\lambda}{2} - 2(\lambda + \frac{1}{2})\beta = 0$$

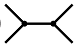
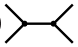
$$\boxed{(\beta + \tfrac{1}{2})(\beta - 1 - \lambda) = 0}\,. \tag{C.33}$$

Combine (C.30) with (C.28):

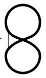
$$\frac{n-7}{2}\lambda = \beta + \frac{1}{2}\,. \tag{C.34}$$

One of those miracles; now just have to check it out for the two solutions of (C.33):

$\underline{\underline{\beta = -\frac{1}{2}}}$ = . Substitute into  contraction:

$$P_A = \frac{(Q-\lambda)A - (\beta - \lambda)}{1/2 - \lambda}$$  $$= \frac{(Q-\lambda)A + (\lambda + 1/2)}{1/2-\lambda}$$ 

(C.35)

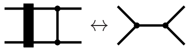
$$d_A = \frac{1}{1/2-\lambda}\left\{-\frac{1}{2}\right.$$  $$\left. - \lambda\frac{n(n-1)}{2} + (\lambda + \frac{1}{2})\right\}$$

$$= \frac{n}{1/2-\lambda}\left\{1 - \lambda\frac{n(n-3)}{2}\right\}$$

$$= \frac{7}{1/2-\lambda}\left\{1 - \lambda \cdot 14\right\} \tag{C.36}$$

$$N = \frac{14(1-14\lambda)}{1-2\lambda}\,. \tag{C.37}$$

There are two subcases: $\underline{\underline{n = 7, \lambda \neq 0}} \Rightarrow \lambda$ indeterminate which means that

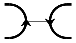
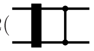
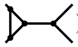
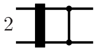
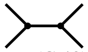


(C.38)

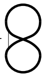
intertangle and

$$\gamma = \frac{1}{2} + \lambda \tag{C.39}$$

$\underline{\underline{\lambda = 0}} \Rightarrow \gamma = \frac{1}{2} \Rightarrow$

$P_A =$  $= \frac{Q}{1/2}P_a = 2($  $-$  $) = 2$  $+$ 

(C.40)

$N = -$  $+$  $= 2n\,.$ (C.41)

G_2 is a solution.
The other solution to (C.33) gives
$\underline{\underline{\beta = 1 + \lambda}} \Rightarrow (n-7)\lambda = 3 + 2\lambda \Rightarrow$

$$\lambda = \frac{3}{n-9}$$
$$\beta = \frac{n-6}{n-9}\,. \tag{C.42}$$

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